7.4, 7.5(a), 8.2(e), 8.4, 9.1(c), 9.2(b), 9.3, 9.4, 9.9(c)

7.4: (1) $a_n = \frac{\sqrt{2}}{n}$, then $a_n \to 0$.

a_n is not rational, since otherwise $n \cdot a_n = \sqrt{2}$ is rational, which is a contradiction.

(2). Let $A_n = \{ r \in \mathbb{Q} \mid \sqrt{2} - \frac{1}{n} < r < \sqrt{2} \}$, for $n \in \mathbb{N}$.

Pick any $a_n \in A_n$, then we get a a sequence (a_n) converges to $\sqrt{2}$.

7.5(a). $\lim_{n\to\infty} S_n = 0$,

$$0 < S_n = \sqrt{n^2+1} - n = \frac{(\sqrt{n^2+1} - n)(\sqrt{n^2+1} + n)}{\sqrt{n^2+1} + n}$$

$$= \frac{(n^2+1) - n^2}{\sqrt{n^2+1} + n} < \frac{1}{n}$$

since $\frac{1}{n} \to 0$. hence $\lim S_n = 0$.

8.2(e) $\lim_{n\to\infty} \frac{1}{n} \sin(n) = 0$

pf: note that $|\sin(n)| \le 1$. Hence, for any $\varepsilon > 0$, $\exists$ integer $N > 0$, s.t. $\frac{1}{N} < \varepsilon$, (by Archimedian property).

Thus, for any $n > N$, we have

$$\left|\frac{1}{n}\sin(n) - 0\right| = \frac{1}{n}\,|\sin(n)| < \frac{1}{n} < \frac{1}{N} < \varepsilon.$$

(8.4) Let (t_n) be bounded, $|t_n| < M$.
and (s_n) converge to 0. Then, show that
$\lim s_n t_n = 0$.

Pf: $\forall\, \varepsilon > 0$, we need to show that $\exists N > 0$
s.t.

$$|s_n t_n - 0| < \varepsilon \qquad \forall\, n > N$$

$$\Leftrightarrow \quad |s_n| \cdot |t_n| < \varepsilon \qquad \forall\, n > N$$

$$\Leftarrow \quad |s_n| < \frac{\varepsilon}{M} \qquad \forall\, n > N.$$

This can be achieved by choosing N large enough, since $s_n \to 0$.

9.1(c).
$$\lim \frac{17n^5 + 73n^4 - 18n^2 + 3}{23n^5 + 13n^3}$$

$$= \lim_{n\to\infty} \frac{17 + 73\cdot\frac{1}{n} - 18\frac{1}{n^3} + \frac{3}{n^5}}{23 + 13\cdot\frac{1}{n^2}}$$

$$= \frac{\lim\left(17 + 73\frac{1}{n} - 18\frac{1}{n^3} + \frac{3}{n^5}\right)}{\lim\left(23 + 13\cdot\frac{1}{n^2}\right)}$$

$$= \frac{\lim 17 + 73 \cdot \lim \frac{1}{n} \cancel{\theta} - 18 \lim \frac{1}{n^3} + 3 \lim \frac{1}{n^5}}{\lim (23) + 13 \cdot \lim \frac{1}{n^2}}$$

$$= \frac{17}{23} \quad \left(\text{since } \lim \frac{1}{n^p} = 0 \quad \forall p > 0\right)$$

9.2(b) $\lim x_n = 3$, $\lim y_n = 7$,

$$\lim_{n \to \infty} \frac{3y_n - x_n}{y_n^2} = ?$$

Pf: since $\lim y_n^2 = (\lim y_n)^2 = 7^2 \neq 0$

we have

$$\lim_{n \to \infty} \frac{3y_n - x_n}{y_n^2} = \frac{\lim 3y_n - \lim x_n}{\lim y_n^2}$$

$$= \frac{3 \cdot 7 - 3}{7^2} = \frac{21 - 3}{49} = \frac{18}{49}$$

9.3 If $\lim x_n = a$, $\lim b_n = b$. and

$$S_n = \frac{a_n^3 + 4a_n}{b_n^2 + 1}$$

Prove that

$$\lim S_n = \frac{a^3 + 4a}{b^2+1}.$$

Pf: we first note that $B_n = b_n^2 + 1 \neq 0$,

and $B = \lim B_n = \lim(b_n^2 + 1)$

$= (\lim b_n)^2 + 1 = b^2 + 1 \neq 0.$

Hence, we may apply the division rule for limit, that

$$\lim S_n = \frac{\lim (a_n^3 + 4a_n)}{\lim (b_n^2 + 1)}$$

$$= \frac{a^3 + 4a}{b^2+1}$$

(9.4) Let $S_1 = 1$, $S_{n+1} = \sqrt{1 + S_n}$

(a) list the first few terms of S_n

(b). Assume that S_n converges. Show that

$$\lim s_n = \frac{1+\sqrt{5}}{2}.$$

Pf: (a) $(s_n) = (1, \sqrt{2}, \sqrt{1+\sqrt{2}}, \sqrt{1+\sqrt{1+\sqrt{2}}}, \cdots$

(b). If s_n converges to $\alpha \in \mathbb{R}$, then

$$s_{n+1}^2 = 1 + s_n.$$

taking limit on both sides, we get

$$\alpha^2 = \alpha + 1$$

$$\Rightarrow \alpha = \frac{1}{2}(1+\sqrt{5}), \text{ or } \frac{1}{2}(1-\sqrt{5}).$$

since $\frac{1}{2}(1-\sqrt{5}) < 0$, and all $s_n > 0$,
we have α can only be $\frac{1}{2}(1+\sqrt{5})$

9.9(c) If $\lim s_n = -\infty$ or $\lim t_n = +\infty$, then there is nothing to prove.
since $+\infty$ is larger than any element in $\mathbb{R} \cup \{\infty, -\infty\}$, and
$-\infty$ is smaller than any element in $\mathbb{R} \cup \{+\infty, -\infty\}$.
If $\lim s_n = +\infty$ or $\lim t_n = -\infty$, then case (a) (b) answers.
Hence, we only need to consider the case where $\lim s_n = \alpha$
$\lim t_n = \beta$, $\alpha, \beta \in \mathbb{R}$. We prove by contradiction, if $\alpha > \beta$,
then let $\varepsilon = \frac{1}{3}(\alpha - \beta)$, hence there exists an $N > 0$. s.t. $\forall n > N$,

$|s_n - \alpha| < \varepsilon$. and $|t_n - \beta| < \varepsilon$. This implies

$\alpha - \varepsilon < s_n < \alpha + \varepsilon$, $\beta - \varepsilon < t_n < \beta + \varepsilon$

since $\alpha + \varepsilon = \alpha + \frac{1}{3}(\alpha - \beta) < \alpha + \frac{2}{3}(\alpha - \beta) = \beta - \frac{1}{3}(\alpha - \beta) = \beta - \varepsilon$.

we have $s_n < t_n \quad \forall n > N$.

This contradict with $s_n < t_n \quad \forall n$. , Hence $\alpha \leq \beta$.