1. Let (s_n) be a bounded seq. Show that

(a) $\liminf s_n \leq \limsup s_n$

(b) $\limsup s_n = \inf \{ \sup_{n \geq N} s_n \mid N \in \mathbb{N} \}$.

Pf: (a) Let $A_N = \sup\{ s_n \mid n \geq N\}$, $B_N = \inf \{ s_n \mid n \geq N\}$.

Then $A_N \geq B_N$. Since (s_n) is bounded, hence $\lim A_N$, and $\lim B_N$ exist, Hence $\lim A_N \geq \lim B_N$ (by Exercise 9.9(c)). Thus $\limsup s_n \geq \liminf s_n$.

(b) Since $A_N \geq A_{N+1}$, and A_N is a bounded sequence, hence $\lim A_N = \inf \{ A_N \mid N \in \mathbb{N}\}$ by Thm 10.2.

We can also prove it directly, let $u = \inf\{A_N \mid N \in \mathbb{N}\}$. Then $A_N \geq u$, $\forall N$. And for any $\varepsilon > 0$, $\exists N$, s.t. $u + \varepsilon \geq A_N$, otherwise $u+\varepsilon$ would be a bigger lower bound of $\{A_N\}$, contradiction with definition of inf. By monotonicity, $\forall n \geq N$, we have

$$u + \varepsilon > A_N \geq A_n \geq u \quad \Rightarrow \quad |A_n - u| < \varepsilon.$$

Hence, A_n converges to u.

2. If (a_n), (b_n) are 2 bounded sequence, show that

$$\limsup (a_n + b_n) \leq \limsup a_n + \limsup b_n.$$

Then give example to illustrate strict inequality.

Pf: We first give example:

$$a_n = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \qquad b_n = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$$

then $a_n + b_n = 1$. Hence

$\limsup (a_n + b_n) = 1$. $\limsup (a_n) = \limsup (b_n) = 1$.

Let $A_N = \sup \{ a_n \mid n \geq N \}$, $B_N = \sup \{ b_n \mid n \geq N \}$

$C_N = \sup \{ a_n + b_n \mid n \geq N \}$. By boundedness of a_n, b_n,

we have A_N, B_N, C_N are all finite, and have limits in $\mathbb{R}$.

We claim that $C_N \leq A_N + B_N$. Given the claim,

we may take limits and get the desired results

$$\lim C_N \leq \lim (A_N + B_N) = \lim A_N + \lim B_N.$$

$$\Leftrightarrow \quad \limsup (a_n + b_n) \leq \limsup a_n + \limsup b_n.$$

Now, to prove the claim, we have $a_n \leq A_N$, $b_n \leq B_N$

$\forall n \geq N$, thus $a_n + b_n \leq A_N + B_N$. Hence

$C_N = \sup \{ a_n + b_n \mid n \geq N \} \leq A_N + B_N$. This gives the claim.

3. (a). Let (s_n) be a seq. s.t.

$$|s_{n+1} - s_n| \leq 2^{-n}. \quad \forall n \in \mathbb{N}.$$

Prove that (s_n) is Cauchy.

(b). If $|s_{n+1} - s_n| \leq \frac{1}{n}$, is the result still true?

Pf: (a) $\forall\, n < m$ integers, we have

$$|s_n - s_m| = \left|\sum_{k=n}^{m-1} s_k - s_{k+1}\right|$$
$$\leq \sum_{k=n}^{m-1} |s_k - s_{k+1}|$$
$$\leq \sum_{k=n}^{m-1} 2^{-k}$$
$$\leq \sum_{k=n}^{\infty} 2^{-k} = 2^{-n+1}.$$

$\forall \varepsilon > 0$, $\exists N > 0$, s.t. $2^{-N} < \varepsilon$. then, $\forall n, m \geq N+1$, we have $|s_n - s_m| \leq 2^{-\min(n,m)+1} \leq 2^{-(N+1)+1} = 2^{-N} < \varepsilon$.

Thus (s_n) is Cauchy

(b) False if we only have $|s_{n+1} - s_n| < \frac{1}{n}$. For example:

$$s_n = 1 + \cdots + \frac{1}{n+1}.$$

then. s_n is monotone increasing and is unbounded.

4

10.7 Let S be a bounded non empty subset in $\mathbb{R}$, s.t. $\sup(S) \notin S$. Show that there is a seq of points (s_n) in S, s.t. $\lim s_n = \sup S$.

Let $u = \sup S$.

Pf: For each $\varepsilon > 0$, $\exists s \in S$, s.t. $u - \varepsilon < s \leq u$. Hence, $\forall n \in \mathbb{N}$, let $\varepsilon = \frac{1}{n}$, we can choose $s_n \in S$, s.t. $u - \frac{1}{n} < s_n \leq u$. Thus, (s_n) forms a sequence in S convergent to u. #

5

10.8 Let S_n be an increasing seq of positive number. define $\sigma_n = \frac{1}{n}(S_1 + \cdots + S_n)$. Then, (σ_n) is an increasing seq.

Pf: $\sigma_{n+1} - \sigma_n = \frac{1}{n+1}(S_1 + \cdots + S_{n+1}) - \frac{1}{n}(S_1 + \cdots + S_n)$

$$= \frac{1}{n(n+1)}\left(n(S_1 + \cdots + S_n) + n \cdot S_{n+1} - (n+1)(S_1 + \cdots + S_n)\right)$$

$$= \frac{1}{n(n+1)}\left((S_{n+1} - S_1) + \cdots + (S_{n+1} - S_n)\right)$$

Since S_n is monotone increasing, then $S_{n+1} - S_k \geq 0$ for all $1 \leq k \leq n$. Hence $\sigma_{n+1} - \sigma_n \geq 0$. #.

6

10.11: Let $t_1 = 1$, $t_{n+1} = \left(1 - \frac{1}{4n^2}\right) \cdot t_n$.

(a) show that $\lim t_n$ exists.

(b). guess what is $\lim t_n$?

Pf: (1) $\because$ $t_{n+1} < t_n$, and $0 < t_n < 1$

$\therefore$ we have a monotone bounded sequence, thus $\lim t_n$ exists,

(2). try computing $\prod_{n=1}^{\infty}\left(1 - \frac{1}{4n^2}\right) = e^{\Sigma \log\left(1 - \frac{1}{4n^2}\right)}$

$$\Sigma_{n=1}^{\infty} \log\left(1 - \frac{1}{4n^2}\right) \approx \Sigma\left(-\frac{1}{4n^2}\right) < \infty$$

Hence, the limit is non-zero.

7 Let $S = \{ x \in (0,1) \mid x = 0.a_1 \cdots a_n, \text{ for some } n, \ a_n = 3 \}$.
Show that for any $t \in (0,1)$, $\exists$ (S_n) in S such that $\lim S_n = t$.

Pf: if $t \in S$, then the constant sequence $S_n = t$ would work. If $t \notin S$, then let

$$t = 0.b_1 b_2 \cdots\cdots$$

be a decimal expansion of t. (if the expansion is finite we add trailing zeros). Then define

$$S_n = 0.b_1 \cdots b_n 3$$

Hence, $|S_n - t| = |0.\underbrace{0 \cdots 0}_{n \text{ many}}3 - 0.\underbrace{0 \cdots 0}_{n \text{ many}} b_{n+1} \cdots|$

$$\leq |0.0 \cdots 03| + |0.\underbrace{0 \cdots 0}_{n \text{ many}} b_{n+1} b_{n+2} \cdots|$$

$$\leq 10^{-n} + 10^{-n}$$

$$= 2 \cdot 10^{-n}.$$

Since $2 \cdot 10^{-n} \to 0$ as $n \to \infty$, we have $S_n \to t$.

#

If you feel uneasy using decimal expansion of a real number, here is an equivalent version:

Let $S_n = \{ 0.a_1 \cdots a_n \mid a_n = 3 \}$. Then $S = \bigsqcup_{n=1}^{\infty} S_n$.

and each S_n is a finite set. Let

$$t_n = \max \{ a \in S_n, \ a \leq t \}$$

then. $t_n \le t \le t_n + 10^{-(n-1)}$

Hence $|t - t_n| \le 10^{-(n-1)}$

Thus. $t_n \to t$.