Math 104, HW #4,

Ross. §11: 5, 10, 11 §12: 10, 11, 12, 13, 14

11.5. Let (q_n) be an enumeration of all rationals in $[0,1]$.

(a) Give the set of subseq limits of (q_n)

(b) Give the values of $\limsup q_n$ and $\liminf (q_n)$.

Solution: (a) The set of the subseq limit S is $[0,1]$.

Indeed, for any $t \in [0,1]$, and any $\varepsilon > 0$,

The set $\{q \mid q \in \mathbb{Q} \cap [0,1] \cap (t-\varepsilon, t+\varepsilon)\}$ is infinite.

Since (q_n) is an enumeration of $\mathbb{Q} \cap [0,1]$, hence the set $\{n \in \mathbb{N} \mid q_n \in (t-\varepsilon, t+\varepsilon)\}$ is infinite. By Thm 11.2, t is a subseq limit of (q_n).

(b) $\limsup (q_n) = \sup S = 1$

$\liminf (q_n) = \inf S = 0$.

11.10 Let (s_n) be the seq of numbers as following

$$\begin{array}{ccccc} 1 & 1/2 & 1/3 & 1/4 & 1/5 \\ 1 & 1/2 & 1/3 & 1/4 & 1/5 \\ 1 & 1/2 & 1/3 & 1/4 & 1/5 \\ 1 & 1/2 & 1/3 & 1/4 & 1/5 \end{array}$$

(a) find all the subseq limits

(b) find limsup and lim inf of (s_n).

Solution: (a) We claim that $S = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is the set of subseq limits.

For each $n \in \mathbb{N}$, there is a subseq of (s_n) with constant value $\frac{1}{n}$, hence $\frac{1}{n}$ is a subseq limit. And $\forall \varepsilon > 0$, $\exists\ n_0 \in \mathbb{N}$, s.t. $\frac{1}{n_0} < \varepsilon$, hence the subset $\{n \mid |s_n - 0| < \varepsilon\}$ contains the indexing subset for the subseq with value $\frac{1}{n_0}$, hence is infinite. By Thm 11.2, there exists a subseq converge to 0. If $t \in \mathbb{R} \setminus S$, then, we can find an $\varepsilon > 0$, s.t. $(t-\varepsilon, t+\varepsilon) \cap S = \emptyset$, in particular, $s_n \notin (t-\varepsilon, t+\varepsilon)$ for any $n \in \mathbb{N}$. Thus S is indeed the sets of subseq limits.

(b) $\limsup s_n = \sup S = 1$, $\liminf s_n = \inf S = 0$.

11.11 Let S be a bounded set. Prove there is an increasing seq (s_n) of points in S, such that $\lim s_n = \sup S$.

Pf: Since S is bounded, hence $\alpha = \sup S$ exists in $\mathbb{R}$. If $\alpha \in S$, then we may take the constant sequence (s_n) with $s_n = \alpha$ $\forall n \in \mathbb{N}$. If $\alpha \notin S$, then by minimality of α among upper bounds of S, for any $\varepsilon > 0$, there exists $x \in S$, such that $\alpha > x > \alpha - \varepsilon$. Hence, the set $A_\varepsilon = \{x \in S \mid \alpha > x > \alpha - \varepsilon\}$ is non empty. Hence, for each $n \in \mathbb{N}$, we may let s_n be any element in $A_{1/n}$. This sequence (s_n) converges to α. We may then take a monotone subseq of (s_n), to achieve the desired requirements.

12.10 Prove (s_n) is bounded if and only if

$$\limsup |s_n| < +\infty.$$

Pf: $\Rightarrow$ If (s_n) is bounded, then $A = \sup\{|s_n| \mid n \in \mathbb{N}\}$ exists, and is greater or equal than $\sup\{|s_n| \mid n > N\}$ for any N. Hence $A \geq \limsup |s_n| \geq 0$, thus $\limsup |s_n|$ exists in $\mathbb{R}$.

$\Leftarrow$ If $\limsup |s_n|$ exists in $\mathbb{R}$, say $\limsup |s_n| = \alpha$.

By definition, $\limsup |s_n| = \lim A_n$, where $A_n = \sup\{|s_m| \mid m \geq n\}$. Hence, for any $\varepsilon > 0$, $\exists N > 0$, s.t. $\forall n > N$, $\alpha - \varepsilon < A_n < \alpha + \varepsilon$. In particular,

$$|s_m| < \alpha + \varepsilon \quad \forall m > N$$

Thus,

$$\sup\{|s_m| \mid m \in \mathbb{N}\} = \sup\{|s_1|, |s_2|, \cdots, |s_N|, |A_{N+1}|\}$$
$$\leq \max\{|s_1|, \cdots, |s_N|, \alpha + \varepsilon\}.$$

which exists in $\mathbb{R}$.

12.11 Prove the first inequality in Thm 12.2. i.e.
Let (s_n) be any seq of nonzero real numbers, then

$$\liminf \left|\frac{s_{n+1}}{s_n}\right| \leq \liminf |s_n|^{1/n}$$

Pf: Consider the possible values of $\alpha = \liminf \left|\frac{s_{n+1}}{s_n}\right|$:

Case $\alpha = 0$: there is nothing to prove, since $|s_n|^{\frac{1}{n}}$ is a seq of positive numbers, hence $\inf\{|s_n|^{\frac{1}{n}} \mid n > N\} \geq 0$, and $\liminf |s_n|^{\frac{1}{n}} \geq 0$.

case $\alpha = +\infty$: we need to show that $\liminf |s_n|^{\frac{1}{n}} = +\infty$, which is equivalent to show $\lim |s_n|^{\frac{1}{n}} = +\infty$. For any $M > 0$, we need to show that there exits an N, s.t. $|s_n|^{\frac{1}{n}} \geq M$ for any $n > N$. Now, since $\liminf |s_{n+1}/s_n| = +\infty$, hence for any M_1, $\exists N_1 > 0$, s.t. $|s_{n+1}/s_n| \geq M_1$ $\forall n > N_1$. Thus, $\forall k \geq 0$.

$$|s_{N_1+k}| = |s_{N_1}| \cdot \left|\frac{s_{N_1+1}}{s_{N_1}}\right| \cdot \left|\frac{s_{N_1+2}}{s_{N_1+1}}\right| \cdot \cdots \left|\frac{s_{N_1+k}}{s_{N_1+k-1}}\right|$$

$$\geq |s_{N_1}| \cdot (M_1)^k = s_N (M_1)^{-N} \cdot (M_1)^{N+k}$$

Hence, $|s_n|^{\frac{1}{n}} \geq C^{\frac{1}{n}} \cdot M_1$. $\forall n > N_1$

where $C = |s_N| \cdot (M_1)^{-N} > 0$.

Since $\lim_{n\to\infty} C^{\frac{1}{n}} = 1$. Hence, $\exists N_2 > 0$. s.t. $\forall n > N_2$, $C^{\frac{1}{n}} > \frac{1}{2}$.

Thus, if we let $M_1 = 2M$, and choose N_1 accordingly, then setting $N = \max(N_1, N_2)$ is enough. Indeed,

$\forall n > N$, $|s_n|^{\frac{1}{n}} \geq C^{\frac{1}{n}} \cdot M_1 \geq \frac{1}{2} \cdot 2M = M$.

This shows $|s_n|^{\frac{1}{n}}$ diverges. to $+\infty$

case $\alpha \in \mathbb{R}$, $\alpha > 0$: the proof is similar to the second case. To show $\liminf |s_n|^{\frac{1}{n}} \geq \alpha$, suffices to show that, for any $1 > \varepsilon > 0$, we have $\liminf |s_n|^{\frac{1}{n}} \geq \alpha(1-\varepsilon)$

Since $\liminf |S_{n+1}/S_n| = \alpha$, hence for any $1 > \varepsilon_1 > 0$, there exists N_1, such that

$$|S_{n+1}/S_n| \geq \alpha(1-\varepsilon_1) \qquad \forall n > N_1$$

$$\Rightarrow \quad |S_{N_1+k}| \geq |S_{N_1}| \cdot (\alpha(1-\varepsilon_1))^k \qquad \forall k > 0$$

$$\Rightarrow \quad |S_n|^{\frac{1}{n}} \geq C^{\frac{1}{n}} \cdot (\alpha(1-\varepsilon_1)) \qquad \forall n > N_1$$

where C is some positive constant, depending on ε_1, N_1.

$$C = |S_{N_1}| \cdot (\alpha(1-\varepsilon_1))^{-N_1}$$

Since $\lim C^{\frac{1}{n}} = 1$, hence $\forall\ 1 > \varepsilon_2 > 0$, $\exists N_2 > 0$, s.t. $1+\varepsilon_2 > C^{\frac{1}{n}} > 1-\varepsilon_2$.

Thus, for $n > \max(N_1, N_2)$, we have

$$|S_n|^{\frac{1}{n}} \geq (1-\varepsilon_2)(1-\varepsilon_1)\cdot\alpha.$$

Thus, for any given $1 > \varepsilon > 0$, if we choose $\varepsilon_1 = \varepsilon_2$, such that $(1-\varepsilon_1)(1-\varepsilon_2) = (1-\varepsilon)$, and if we set $N = \max(N_1, N_2)$, where N_1, N_2 are obtained as above, then

$$\forall n > N, \qquad |S_n|^{\frac{1}{n}} \geq \alpha(1-\varepsilon).$$

This shows. $\liminf |S_n|^{\frac{1}{n}} \geq \alpha$.

12.12 Let (S_n) be a sequence of non-negative numbers, and for each n, define

$$\sigma_n = \frac{1}{n}(S_1 + \cdots + S_n).$$

(a). Show that

$$\liminf S_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup S_n$$

(b) Show that, if $\lim s_n$ exists, then $\lim \sigma_n$ exists and equals $\lim s_n$.

(c) Given an example where $\lim \sigma_n$ exists, but $\lim s_n$ doesn't.

a short but incomplete proof.

<u>Solution</u>: (a) We may prove this directly, or we may cheat by considering sequence

$$a_n = e^{s_1 + \cdots + s_n}$$

Then $a_{n+1}/a_n = e^{s_{n+1}}$, $(a_n)^{\frac{1}{n}} = e^{\sigma_n}$,

from thm 12.2,

$$(*) \quad \liminf (e^{s_n}) \le \liminf (e^{\sigma_n}) \le \limsup (e^{\sigma_n}) \le \limsup (e^{s_n})$$

<u>Lemma</u>: if (b_n) is a sequence of non-negative real number. then

$$\liminf (e^{b_n}) = e^{\liminf b_n}$$

$$\limsup (e^{b_n}) = e^{\limsup b_n}.$$

where we set $e^{+\infty} = +\infty$.

However, to prove this lemma needs some effort. Hence if you take this for granted, then

$$(*) \Rightarrow \quad e^{\liminf (s_n)} \le e^{\liminf \sigma_n} \le e^{\limsup \sigma_n} \le e^{\limsup s_n}$$

$$\Leftrightarrow \quad \liminf (s_n) \le \liminf (\sigma_n) \le \limsup \sigma_n \le \limsup (s_n).$$

(a) Now we prove this in the style of Thm 12.2.

We first prove that

(1) $$\limsup \sigma_n \le \limsup s_n.$$

If $\limsup s_n = +\infty$, then there is nothing to prove.

Now, assume $\alpha = \limsup s_n \in \mathbb{R}$. Hence we only need to show that, $\forall \varepsilon > 0$.

$$\limsup \sigma_n \le \alpha + \varepsilon.$$

Since $\alpha = \limsup s_n$, thus for any $\varepsilon_1 > 0$, $\exists N_1 > 0$, s.t.

$$\alpha + \varepsilon_1 > s_n \qquad \forall n > N_1$$

Hence
$$(N_1+k)\,\sigma_{N_1+k} - N_1 \sigma_{N_1} = s_{N_1+1} + \cdots + s_{N_1+k}$$
$$\le (\alpha + \varepsilon_1)\cdot k$$

$$\therefore \quad \sigma_{N_1+k} \le \frac{N_1}{N_1+k}\sigma_{N_1} + \frac{k}{N_1+k}(\alpha+\varepsilon_1).$$

taking limsup over k on both sides, we get

$$\limsup_{n\to\infty} \sigma_n = \limsup_{k\to\infty} \sigma_{N_1+k} \le \limsup\left(\frac{N_1}{N_1+k}\sigma_{N_1} + \frac{k}{N_1+k}(\alpha+\varepsilon_1)\right)$$
$$= \alpha + \varepsilon_1.$$

If we choose $\varepsilon_1 = \varepsilon$, we get the desired result,

The claim about $\liminf s_n \le \liminf \sigma_n$ is similar.

(b) $\lim s_n$ exists $\Rightarrow \liminf \sigma_n = \limsup \sigma_n = \lim s_n$

$\Rightarrow \lim \sigma_n = \lim s_n.$

(c). Let $s_n = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd,} \end{cases}$ then $\sigma_n \to \frac{1}{2}$

but (s_n) doesn't. converge.

#12.13. Let (S_n) be a bounded seq in $\mathbb{R}$. If

$A = \{a \in \mathbb{R} \mid \text{only finitely many } S_n < a\}$.

$B = \{b \in \mathbb{R} \mid \text{only finitely many } S_n > b\}$.

then $\sup A = \liminf S_n$, $\inf B = \limsup a_n$.

Pf: We prove only one side, $\sup(A) = \liminf S_n$.

For each $n > 0$ integer, let

$$\alpha_n = \inf\{S_m \mid m \geq n\}.$$

then by definition, $S_m \geq \alpha_n$ $\forall m \geq n$, hence

$\{m \mid S_m < \alpha_n\} \subset \{1, 2, \cdots, n\}$ is finite. hence $\alpha_n \in A$.

Let's denote $A' = \{\alpha_n \mid n \in \mathbb{N}\}$. then $A' \subset A$.

Next, we show that $\forall a \in A$, $\exists \alpha_n \geq a$. Indeed,

if $a \in A$, then $\{n \mid S_n < a\}$ is finite, and let N

$= \max\{n \mid S_n < a\}$. We then have $S_n \geq a$ $\forall n > N$,

hence $a \leq \inf\{S_m \mid m > N\} = \alpha_N$. This shows

$$\sup(A) = \sup(A') = \liminf S_n.$$

12.14 (a) $\lim (n!)^{\frac{1}{n}}$.

Let $S_n = n!$, then $S_{n+1}/S_n = n+1$,

hence $\lim(S_{n+1}/S_n) = +\infty$

$\lim(n!)^{\frac{1}{n}} = \lim (S_n)^{\frac{1}{n}} =$

(b) $\lim \frac{1}{n}(n!)^{\frac{1}{n}} = \lim \left(\frac{n!}{n^n}\right)^{\frac{1}{n}}$

let $S_n = n!/n^n$, then

$$S_{n+1}/S_n = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = (n+1) \cdot \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1} = \frac{1}{(1+\frac{1}{n})^n}$$

we have $\lim (1+\frac{1}{n})^n = e$, hence.

$$\lim (S_{n+1}/S_n) = \frac{1}{e}.$$ thus

$$\lim \frac{1}{n}(n!)^{\frac{1}{n}} = \frac{1}{e}.$$