Math 104. HW 6. Rudin 2.11, 2.14, 2.15, Ross 14.14,

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11. For $x \in R^1$ and $y \in R^1$, define

$$d_1(x, y) = (x-y)^2,$$
$$d_2(x, y) = \sqrt{|x-y|},$$
$$d_3(x, y) = |x^2 - y^2|,$$
$$d_4(x, y) = |x - 2y|,$$
$$d_5(x, y) = \frac{|x-y|}{1+|x-y|}.$$

Determine, for each of these, whether it is a metric or not.

(1) $d_1(x,y) = (x-y)^2$ is not a metric, because it fails triangle inequality. Say $x=0$, $y=1$, $z=2$, then.

$d(x,y) = 1$, $d(y,z) = 1$, $d(x,z) = (0-2)^2 = 4$

and $d(x,y) + d(y,z) < d(x,z)$.

(2) $d_2(x-y) = \sqrt{|x-y|}$ is a metric. The conditions are obvious except triangle inequality, we need to check that

(*) $\sqrt{|x-y|} + \sqrt{|y-z|} \geq \sqrt{|x-z|} \quad \forall x,y,z \in \mathbb{R}$

(*) $\Leftrightarrow (\sqrt{|x-y|} + \sqrt{|y-z|})^2 \geq |x-z|$

$\Leftrightarrow |x-y| + |y-z| + 2\sqrt{|x-y|}\cdot\sqrt{|y-z|} \geq |x-z|$

$\Leftarrow \begin{cases} 2\sqrt{|x-y|}\cdot\sqrt{|y-z|} \geq 0 \\ |x-y| + |y-z| \geq |x-z|. \end{cases}$

Hence (*) is satisfied.

(c). $d_3(x,y) = |x^2-y^2|$ is not a metric, since we can have $x \neq y$, but $d_3(x,y) = 0$. e.g. $x=1, y=-1$.

(d) $d_4(x,y) = |x-2y|$ is not symmetric in x,y.

i.e. $\exists x,y \in \mathbb{R}$, s.t. $d_4(x,y) \neq d_4(y,x)$ For example,

$$d_t(1,0) = 1, \qquad d_t(0,1) = 2.$$

(e) $d_5(x,y) = \dfrac{|x-y|}{1+|x-y|}$ is a metric. We only need to check the triangle inequality, since the other is obvious.

Let $f(r) = \dfrac{r}{1+r}$, where $r \geq 0$. Then we claim that if $0 \leq a < b$, then $f(a) < f(b)$. Indeed, we have.

$$\frac{a}{1+a} < \frac{b}{1+b} \iff a(1+b) < b(1+a) \iff a+ab < b+ab$$

$$\iff a < b.$$

Next, we claim that if $a \geq 0$, $b > 0$, then.

$$f(a) + f(b) \geq f(a+b).$$

$$\frac{a}{1+a} + \frac{b}{1+b} \geq \frac{a+b}{1+a+b}$$

$$\iff \frac{a(1+b) + b(1+a)}{(1+a)(1+b)} \geq \frac{a+b}{1+a+b}.$$

$$\iff \frac{a+b+2ab}{1+a+b+ab} \geq \frac{a+b}{1+a+b}.$$

Given any 4 positive real number A, B, C, D, if $\frac{A}{B} \leq \frac{C}{D}$, then

$$\frac{A}{B} \leq \frac{A+C}{B+D} \leq \frac{C}{D}.$$

Indeed, $A(B+D) \leq B(A+C)$, and. $(A+C)D \leq C \cdot (B+D)$.

Let $A = a+b$, $B = 1+a+b$, $C = 2ab$, $D = ab$. , we have.

$\frac{A}{B} < 1 < 2 = \frac{C}{D}$, hence

$$\frac{a+b+2ab}{1+a+b+ab} \geq \frac{a+b}{1+a+b}.$$

14. Give an example of an open cover of segment $(0,1)$ which has no finite sub cover.

Sol: Let $(a_n)_{n=1}^{\infty}$ be a sequence of real number, such that $a_n \in (0,1)$, $a_n > a_{n+1}$, and $\lim a_n = 0$. Then define $U_n = (a_n, 1)$. We then have $U_1 \subset U_2 \subset U_3 \subset \cdots$, and $\bigcup_n U_n = (0,1)$. Indeed, for any $x \in (0,1)$, there exist N large enough, s.t. $\forall n > N$, $a_n < x$. thus, $x \in (a_n, 1)$ for all $n > N$, hence $x \in \bigcup_n U_n$.

Now, we show $\bigcup_{n \in \mathbb{N}} U_n$ does not have a finite subcover for $(0,1)$. Indeed, for any finite subset $I \subset \mathbb{N}$, let $n' = \max(I)$, then $\bigcup_{n \in I} U_n = U_{n'} = (a_{n'}, 1) \neq (0,1)$.

[Many other possible covers, e.g. (set $a_0 = 1$), and let

$$U_n = (a_{n+1}, a_{n-1})$$

2.15 Show that thm 2.36 and its corollary become false, if we replace "compact" by "bounded", or by "closed"

2.36 Theorem *If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.*

Proof Fix a member K_1 of $\{K_\alpha\}$ and put $G_\alpha = K_\alpha^c$. Assume that no point of K_1 belongs to every K_α. Then the sets G_α form an open cover of K_1; and since K_1 is compact, there are finitely many indices $\alpha_1, \ldots, \alpha_n$ such that $K_1 \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$. But this means that

$$K_1 \cap K_{\alpha_1} \cap \cdots \cap K_{\alpha_n}$$

is empty, in contradiction to our hypothesis.

Corollary *If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$ $(n = 1, 2, 3, \ldots)$, then $\bigcap_1^\infty K_n$ is not empty.*

Take $X = \mathbb{R}^1$.

Sol'n: (1) if we replace compact by closed, then, we may let $K_n = [n, +\infty)$, then $\bigcap K_n = \phi$.

(2) if we replace "compact" by "bounded", we may let $K_n = (0, \frac{1}{n})$.

optional: if we replace compact by closed and bounded, then, if we let $X = \mathbb{Q}$, and let $K_n = [\sqrt{2} - \frac{1}{n}, \sqrt{2}] \cap \mathbb{Q}$

K_n is closed in X (not closed in $\mathbb{R}$)., and is bounded, but K_n is not compact. $\bigcap K_n = \phi$.

Ross 14.14:

Show that $\sum_{n=1}^{\infty} \frac{1}{n} \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \cdots$

Hence $\sum \frac{1}{n}$ diverges to $+\infty$.

Pf: We decompose $\mathbb{N}$ into disjoint intervals,

$I_0 = \{1\}$, $I_m = [2^{m-1}+1, 2^m] \cap \mathbb{N}$ for $m \geq 1$

then for $n \in I_m$, we let $a_n = 2^{-m}$.

$$\sum_{n \in I_m} a_n = (2^m - 2^{m-1}) \cdot 2^{-m} = \frac{1}{2}. \quad \forall m \geq 1.$$

Hence, the partial sum of a_n is.

$$S_{2^m} = \sum_{n=1}^{2^m} = \sum_{n \in I_0} a_n + \sum_{n \in I_1} a_n + \cdots + \sum_{n \in I_m} a_n$$

$$= 1 + \frac{1}{2} + \cdots + \frac{1}{2} = 1 + \frac{m}{2}.$$

m

And $\lim_{m\to\infty} S_{2^m} = +\infty$.

On the other hand, $b_n = \frac{1}{n}$, if $n \in I_m$, then $n \leq 2^m$, thus $\frac{1}{n} \geq \frac{1}{2^m} = a_n$. Hence $b_n \geq a_n$. By comparison test, $\sum b_n = +\infty$.

#5 If $a_n > 0$ and $\sum a_n$ diverges, then $\sum \frac{a_n}{1+a_n}$ diverges.

Pf: If $\sum \frac{a_n}{1+a_n}$ converges, then $b_n = \frac{a_n}{1+a_n} \to 0$, hence $a_n = \frac{b_n}{1-b_n} \to 0$. Let N be large enough, s.t. $\forall n \geq N$, $a_n < 1$, then $\sum_{n=N}^{\infty} \frac{a_n}{1+a_n} > \sum_{n=N}^{\infty} \frac{a_n}{2} = \frac{1}{2} \sum_{n=N}^{\infty} a_n$.

Hence $\sum a_n$ converges. Contradict with the hypothesis.

#6 If $a_n > 0$ and $\sum a_n$ converges, show that $\sum \sqrt{a_n}/n$ converges.

Pf: let $A_n = \sum_{j=1}^n a_j$, $B_n = \sum_{j=1}^n \sqrt{a_j}/j$, be partial sums.

Then

$$B_n^2 = \left(\sum_{j=1}^n \sqrt{a_j} \cdot \frac{1}{j}\right)^2 \leq \left(\sum_{j=1}^n (\sqrt{a_j})^2\right) \cdot \left(\sum_{j=1}^n \frac{1}{j^2}\right)$$

$$= \left(\sum_{j=1}^n a_j\right) \cdot \left(\sum_{j=1}^n \frac{1}{j^2}\right)$$

Since $\sum a_n$ converges and $\sum \frac{1}{n^2}$ converges, say to limit

S and T respectively, hence, $\forall n$.

$$\sum_{j=1}^{n} a_j \leq S \qquad \text{and} \qquad \sum_{j=1}^{n} \frac{1}{j^2} \leq T.$$

Thus $B_n^2 \leq T \cdot S$. Since B_n is a monotone increasing sequence and is bounded, hence B_n converges.