1. If a_{nm} is a double seq in $\mathbb{R}$, such that
$$\lim_n a_{nm} = 0, \qquad \lim_m a_{nm} = 1.$$
Is it true that $\lim_n a_{nn}$ exists and is in $[0,1]$?

Ans: No. For any given (a_{nm}) with the above condition, we can define
$$b_{nm} = \begin{cases} a_{nm} & n \neq m \\ C_n & n = m \end{cases}$$
where C_n is any sequence, say constant sequence (-1), then b_{nm} satisfies the same condition as a_{nm}, but $\lim_n b_{nn} = \lim_n C_n$ can be anything.

2. If $f: X \to \mathbb{R}$ is Lipschitz continuous, then it is uniform continuous.

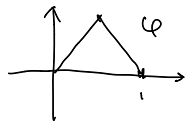
Pf: By Lipschitz continuity of f, we know there exists $K > 0$, s.t. $|f(x) - f(y)| \leq K \cdot d(x,y)$. Hence, $\forall \varepsilon > 0$, we may choose $\delta = \frac{\varepsilon}{K}$, so that for any $x, y \in X$, with $d(x,y) < \delta$, we have $|f(x) - f(y)| \leq K \cdot d(x,y) < K \cdot \delta = \varepsilon$..

3. Let $f_n : X \to \mathbb{R}$ be a sequence of Lipschitz continuous function, with the same Liptschiz constant. Assume that $f_n \to f$ pointwise.
Is it true that $f_n \to f$ uniformly ?

Ans: No. Two counter examples:

Let $\varphi(x)$ be a $\overset{\text{Lipschitz}}{\text{bump}}$ function supported on $[0,1]$.

e.g.



Let $f_n(x) = \varphi(x-n)$. Then $\forall x \in \mathbb{R}$, $f_n(x) \to 0$ as $n \to 0$. , and $f_n(x)$ are Lipschitz with the same constants. But $f_n(x)$ does not converge to 0 uniformly.

Ex2: $f_n(x) = \frac{x}{n} : \mathbb{R} \to \mathbb{R}$. Then $\forall x \in \mathbb{R}$, $f_n(x) \to 0$ as $n \to \infty$.

and $f_n(x)$ are all Liptschitz with constant $K = 1$.

However. $f_n(x)$ do not converge to 0 uniformly over $\mathbb{R}$.

(credit: Anton Thai)

4. Prove that $f_n(x) = \frac{\sin(x)}{1+nx^2}$ converges uniformly on $\mathbb{R}$.

Pf: We will prove that $f_n(x)$ converges to 0 uniformly on (a) $|x| \leq 2$ (b) $|x| \geq 1$.

(a) For $x \in [-2,2]$, consider $g_n(x) = |f_n(x)| = \frac{|\sin(x)|}{1+n\cdot x^2}$.

then (1) $\forall x \in [-2,2]$, $g_n(x) \to 0$ as $n \to \infty$.

(2) $\forall x \in [-2,2]$, $g_n(x) \geq g_{n+1}(x)$.

(3) $[-2,2]$ is a compact set and each $g_n(x)$ is continuous

Thus, by Rudin Thm 7.13, we see $g_n \to 0$ uniformly over $[-2,2]$.

(b) For $|x| \geq 1$, we have $|f_n(x)| \leq \frac{1}{1+n\cdot x^2} \leq \frac{1}{1+n}$. , hence $f_n \to 0$ uniformly over $\{x \in \mathbb{R} \mid |x| \geq 1\}$.

5. Let $f_n, g_n : X \to \mathbb{R}$ be sequences of continuous functions. Suppose $f_n \to f$ and $g_n \to g$ uniformly. is it true that $f_n \cdot g_n \to f \cdot g$ uniformly ?

Ans : No. We may write $f_n(x) = f(x) + \alpha_n(x)$, $g_n(x) = g(x) + \beta_n(x)$. then $\alpha_n \to 0$, $\beta_n \to 0$ uniformly. But

$$f_n \cdot g_n = (f + \alpha_n)(g + \beta_n) = f \cdot g + f \cdot \beta_n + \alpha_n \cdot g + \alpha_n \cdot \beta_n.$$

The problem is that, $f \beta_n$ or $g \cdot \alpha_n$ may not converge to 0 uniformly.

For example : $f_n(x) = x$, $g_n(x) = \frac{1}{n}$.. Then their uniform limits are $f(x) = x$ and $g(x) = 0$. However $f_n g_n = \frac{1}{n} x$ do not converge to $fg = 0$ uniformly.

Extra thoughts : in problem 3 and 5, if we restrict the domain X. to be compact set, then the answer would be true in both case.

Problem 3 (assuming X compact). We first prove that $f(x)$ is Lipschitz continuous with the same Liptschiz constant K. Indeed, for any $x, y \in X$, we have

$$|f(x) - f(y)| = \lim_{n \to \infty} |f_n(x) - f_n(y)| \leq K \cdot d(x,y).$$

Let $g_n(x) = f_n(x) - f(x)$, then $\{g_n(x)\}$ are Liptschitz continuous.
with constant $2K$, since $\forall x, y \in X$.

$$|g_n(x) - g_n(y)| \leq |f_n(x) - f_n(y)| + |f(x) - f(y)| \leq 2K \cdot d(x,y).$$

and $g_n(x) \to 0 \quad \forall x \in X$. We only need to show that $g_n \to 0$ uniformly. For any $\varepsilon > 0$, let $r = \frac{\varepsilon}{3K}$. Then by compactness of X, the cover $X = \bigcup_{x \in X} B_r(x)$ exists a finite subcover

$$X = B_r(x_1) \cup \cdots \cup B_r(x_M)$$

for some finite subset $\{x_1, x_2, \cdots, x_M\} \subset X$. For each $i \in \{1, \cdots, M\}$, since $g_n(x_i) \to 0$, there exists N_i, such that $\forall n > N_i$, $|g_n(x_i)| < \frac{\varepsilon}{3}$. Let $N = \max\{N_1, \cdots, N_M\}$. Then $\forall n > N$, $\forall y \in X$, there exists a ball $B_r(x_i) \ni y$, hence

$$|g_n(x) - g_n(y)| \leq 2K \cdot d(x,y) = 2K \cdot \frac{\varepsilon}{3K} = \frac{2}{3}\varepsilon.$$

and $$|g_n(y)| \leq |g_n(x)| + |g_n(x) - g_n(y)| \leq \frac{\varepsilon}{3} + \frac{2}{3}\varepsilon = \varepsilon.$$

This shows the uniform convergence of g_n to 0.

<u>Problem 5</u>: X compact $\Rightarrow$ f and g are bounded, say $|f| \leq M$, $|g| \leq M$,

then $|f \cdot \beta_n| \leq M \cdot |\beta_n|$, $|g \cdot \alpha_n| \leq M \cdot |\alpha_n|$

and we have uniform convergence $f\beta_n \rightrightarrows 0$, $g \cdot \alpha_n \rightrightarrows 0$.