Math 104 Midterm 1 Apr 1st, 2021 (B)

Name: ______________________________

- Please write your name, your section, the time slot that you take the exam on the first page of your submission.
- The midterm is 80 minutes. This is a closed book exam. You can take one cheatsheet with you. After the exam ends, you have 10 minutes to scan and upload your exam to gradescope.
- You can also use paper or iPad to write your exam. Latex is also OK.
- You can use theorems and results (in example and exercises) in the textbook, unless the problem requires you to prove from the definition.
- No calculator should be used. No discussion or searches on internet are allowed.
- If you have question during the exam, you may contact me using zoom direct message or via email.

Good Luck!

Question	Points	Score
1	20	
2	20	
3	15	
4	15	
5	15	
6	15	
Total	100	

1. (20 points, 4 points each) Topology on $\mathbb{Q}$. You may use the fact that the topology on $\mathbb{Q}$ constructed from the induced metric on $\mathbb{Q}$ as a subset of $\mathbb{R}$ is equivalent to the induced topology on $\mathbb{Q}$ as a subset of $\mathbb{R}$.

   Answer the following question and **justify your answer.**

   (1) Is the set $(-\sqrt{3}, \sqrt{3}) \cap \mathbb{Q}$ open in $\mathbb{Q}$? Is it closed in $\mathbb{Q}$?

   (2) Is the set $(-1, 1) \cap \mathbb{Q}$ open in $\mathbb{Q}$? Is it closed in $\mathbb{Q}$?

   (3) Is the set $(0, 1) \cap \mathbb{Q}$ connected?

   (4) Find a subset $K \subset \mathbb{Q}$, such that K is closed and bounded in $\mathbb{Q}$, but not compact.

   (5) Find a subset $K \subset \mathbb{Q}$, such that K is compact and K is an infinite set.

2. (20 points, 2 each) True or False. **No justification is needed.**

   Let $f : X \to Y$ be a continuous map between metric spaces. Let $A \subset X$ and $B \subset Y$.

   (1) If B is open, then $f^{-1}(B)$ is open.

   (2) If B is closed, then $f^{-1}(B)$ is closed.

   (3) If B is bounded, then $f^{-1}(B)$ is bounded.

   (4) If B is connected, then $f^{-1}(B)$ is connected.

   (5) If B is compact, then $f^{-1}(B)$ is compact.

   (6) If A is open, then $f(A)$ is open.

   (7) If A is closed, then $f(A)$ is closed.

   (8) If A is bounded, then $f(A)$ is bounded.

   (9) If A is connected, then $f(A)$ is connected.

   (10) If A is compact, then $f(A)$ is compact.

3. (15 points) Prove that the finite union of compact subsets is compact. More precisely, let X be a metric space, and let $K_1, \cdots, K_n$ be any compact subsets of X. Show that $K_1 \cup \cdots \cup K_n$ is a compact subset of X.

4. (15 points) Let $g : \mathbb{R} \to \mathbb{Q}$ be a continuous map. Prove that g is a constant map, i.e., there exists a constant $\alpha \in \mathbb{Q}$ such that $g(x) = \alpha$ for all $x \in \mathbb{R}$.

5. (15 points) Let $f : \mathbb{Q} \to \mathbb{R}$ be a continuous map. Is it true that one can always find a continuous map $g : \mathbb{R} \to \mathbb{R}$ extending f, namely, $g(x) = f(x)$ for any $x \in \mathbb{Q}$? If true, prove it; if false, give a counter-example and prove that no such extension is possible.

6. (15 points) For each $N \in \mathbb{N}$, let $f_N(x) = \sum_{n=0}^{N} 3^{-n} \sin(2^n x + n)$ be a real valued function on $\mathbb{R}$. Prove that as $N \to \infty$, f_N converges uniformly to some real valued continuous function.