Math 104 Final May 12-13, 2021

Name: ____________________

- Please write your name, your section, the time slot that you take the exam on the first page of your submission.
- The final is 180 minutes. This is a closed book exam. You can take two cheatsheets (4 sided) with you. After the exam ends, you have 10 minutes to scan and upload your exam to gradescope.
- You can also use paper or iPad to write your exam. Latex is also OK.
- You can use theorems and results (in example and exercises) in the textbook, unless the problem requires you to prove from the definition.
- No calculator should be used. No discussion or searches on internet are allowed.
- If you have question during the exam, you may contact me using zoom direct message or via email.

Good Luck!

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

1. True or False (no justification is needed) (2 points each)

F (1) Let (x_n) be a sequence in $[0,1]$, then there is a subsequence of (x_n) whose limit is $\inf\{x_n \mid n \in \mathbb{N}\}$.

T (2) Let $(x_n), (y_n)$ be two bounded sequences of positive real numbers, then

$$\limsup_{n\to\infty}(x_n y_n) \le \limsup_{n\to\infty}(x_n) \limsup_{n\to\infty}(y_n)$$

F (3) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$, then there is a $\delta > 0$, such that f is continuous on $(a-\delta, a+\delta)$.

F (4) If $f : [0,1] \to \mathbb{R}$ is Riemann integrable, then the function $F(x) = \int_0^x f(t)dt$ exists and $F'(x) = f(x)$ for all $x \in (0,1)$.

F (5) If $f : \mathbb{R} \to \mathbb{R}$ is a smooth function, then for any $a \in \mathbb{R}$ and $x \in \mathbb{R}$, we have convergence

$$\lim_{N\to\infty} \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(x).$$

2. True or False? If true, give a proof, if false, give a counter-example (5 points each)

(a) Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} (-1)^k f\left(\frac{k}{n}\right) = 0$$

✓

(a)
f is uniformly cont.
$\forall \varepsilon > 0$, $\exists \delta$ s.t. if $|x-y| < \delta$, $|f(x)-f(y)| < \varepsilon$.
If $\frac{1}{n} < \delta$, then $\left|\sum_k (-1)^k f\left(\frac{k}{n}\right)\right| \le \frac{n}{2} \cdot \varepsilon$.

(b) Let $f : (0,\infty) \to (0,\infty)$ be uniformly continuous. Then

$$\lim_{x\to\infty} \frac{f(x+1/x)}{f(x)} = 1.$$

✗

(b) e.g. $f(x) = e^{-x^2}$.

$$\frac{f(x+\frac{1}{x})}{f(x)} = \frac{e^{-(x+\frac{1}{x})^2}}{e^{-x^2}} = e^{-2-\frac{1}{x^2}} \to e^{-2}$$

3. Calculate and give your justifications.

(a) (3 points)

$$\lim_{x\to 0} \frac{a^x - 1}{x}, \quad a > 0$$

L'Hopital. $\underline{\ln a}$

(b) (3 points)

$$0 \le \lim_{n\to\infty} \int_0^1 \frac{x^n}{1+x} dx \le \lim \int_0^1 x^n\, dx = \lim \frac{1}{n+1} = 0$$

(c) (4 points) For any integer $k \ge 0$, calculate

$$\lim_{n\to\infty} \left(\frac{1^k + 3^k + \cdots (2n-1)^k}{n^{k+1}} \right)$$

$f(x) = x^k$

$$\frac{1}{n} \sum_{m=1}^{n} \left(\frac{2m-1}{n}\right)^k = \frac{1}{2} \cdot \sum_{m=1}^{n} f\left(\frac{2m-1}{n}\right) \cdot \left(\frac{2}{n}\right)$$

0 2

$$\xrightarrow{n\to\infty} \frac{1}{2} \int_0^2 f(x)\, dx$$

$$= \frac{1}{2} \left. \frac{x^{k+1}}{k+1} \right|_{x=0}^{x=2} = \frac{1}{2} \cdot \frac{2^{k+1}}{k+1} = \frac{2^k}{k+1}$$

4. (10 points) Define the sequences (a_n) and (b_n) as follows:

$$0 < b_1 < a_1, \quad a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}$$

Show that (a_n) and (b_n) both converge to the same limit.

Show $a_n \searrow$, $b_n \nearrow$, bounded

$a_n > b_n \quad \forall n.$

Thus a_n, b_n converges.

Let $A = \lim a_n$, $B = \lim b_n$.

$A = \frac{A+B}{2} \Rightarrow A = B$

5. Find the radius of convergence of the following power series on $\mathbb{R}$

$$\sum_{n=0}^{\infty} (2 + (-1)^n)^n x^n$$

$\limsup \left[(2+(-1)^n)^n \right]^{\frac{1}{n}} = 3.$

$\Rightarrow R = \frac{1}{3}$

6. (10 points) Show the following series of function converge uniformly on $\mathbb{R}$

$$\sum_{n=1}^{\infty} n^2 x^2 e^{-n^2|x|}$$

#6. $\cdot$ $\sup_{x \in \mathbb{R}} (x^2 e^{-|x|}) = M.$

$\cdot$ $n^2 x^2 e^{-n^2|x|} = \frac{1}{n^2} (n^2 x)^2 e^{-n^2|x|}$

$\leq \frac{1}{n^2} \cdot M.$

$\because \sum \frac{1}{n^2} < \infty$

$\therefore$ by M-test .....

7. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Prove that, for any $x_1, \cdots, x_n \in \mathbb{R}$, there exists x_0, such that

$$f(x_0) = \frac{1}{n}(f(x_1) + \cdots + f(x_n))$$

#7 Let x_i realize $\max\{f(x_1), \cdots, f(x_n)\}$

x_j realize $\min\{\cdots\}$

then by intermediate value thm

$\exists x_0$ between x_i, x_j, s.t.

$f(x_0) = \frac{1}{n}(f(x_1) \cdots f(x_n))$

8. Let (X, d) be a metric space and $p \in X$ is a fixed point. For any $u \in X$, we define a function $f_u(x) = d(u, x) - d(p, x)$. Show that (5 points each)
(a) For any $u \in X$, the function f_u is bounded.
(b) For any $u, v \in X$, $d(u, v) = \sup_{x \in X} |f_u(x) - f_v(x)|$.

#8 (a) $d(u,x) - d(p,x) \leq d(u,p)$

$d(p,x) - d(u,x) \leq d(u,p)$

$\Rightarrow |d(u,x) - d(p,x)| \leq d(u,p)$

(b) $|f_u(x) - f_v(x)| \leq d(u,v)$

equal sign can be achieved for $x = u$ or v.

9. Show that (5 points each)
(a) $x \geq \sin(x)$, for $x > 0$
(b) $1 - x^2/2 \leq \cos(x)$, for $x > 0$ (you may use result from (a))

Hint: Use mean value theorem.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $f(x)$ be differentiable at $x = a$. Let $\{x_n\}, \{y_n\}$ be two sequences converging to a, such that $x_n < a < y_n$. Prove that

$$f'(a) = \lim_{n \to \infty} \frac{f(y_n) - f(x_n)}{y_n - x_n}$$

#9 (a) $f(x) = x - \sin x.$

$f'(x) = 1 - \cos x.$ $\quad \because \forall x > 0,\ f'(x) > 0$, and $f(0) = 0$

$\therefore f(x) = \int_0^x f'(t)\, dt > 0. \quad \forall x > 0.$

(b) $g(x) = \cos(x) - (1 - \frac{x^2}{2}).$

$g'(x) = -\sin x + x$. $\because \forall x > 0.\ g'(x) > 0$ (by part a). $\quad g(0) = 0.$

$\therefore g(x) = \int_0^x g'(t)\, dt > 0. \quad \forall x > 0$

#10 Since $f(x)$ is differentiable at $x = a$, thus

$h(x) := \frac{f(x) - f(a)}{x - a} - f'(a)$

converges to 0, as $x \to a$. $f(x) = f(a) + f'(a)(x-a) + h(x)(x-a)$

$$\frac{f(y_n) - f(x_n)}{y_n - x_n} = \frac{f(a) + f'(a)(y_n - a) + h(y_n)(y_n - a) - [f(a) + f'(a)(x_n - a) + h(x_n)(x_n - a)]}{y_n - x_n}$$

$$= f'(a) + h(y_n) \cdot \frac{y_n - a}{y_n - x_n} + h(x_n) \frac{a - x_n}{y_n - x_n}.$$

since $0 < \frac{y_n - a}{y_n - x_n} < 1$, and $0 < \frac{a - x_n}{y_n - x_n} < 1$,

hence $\lim_{n \to \infty} h(y_n) \cdot \frac{y_n - a}{y_n - x_n} = 0$, $\lim_{n \to \infty} h(x_n) \frac{a - x_n}{y_n - x_n} = 0$

Thus $\lim_{n \to \infty} \frac{f(y_n) - f(x_n)}{y_n - x_n} = f'(a)$.