Math 104 Final May 12-13, 2021

Name: ______________________________

- Please write your name, your section, the time slot that you take the exam on the first page of your submission.
- The final is 180 minutes. This is a closed book exam. You can take two cheatsheets (4 sided) with you. After the exam ends, you have 10 minutes to scan and upload your exam to gradescope.
- You can also use paper or iPad to write your exam. Latex is also OK.
- You can use theorems and results (in example and exercises) in the textbook, unless the problem requires you to prove from the definition.
- No calculator should be used. No discussion or searches on internet are allowed.
- If you have question during the exam, you may contact me using zoom direct message or via email.

Good Luck!

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

1. True or False (no justification is needed) (2 points each)

   (1) Let (x_n) be a sequence in $[0,1]$, then there is a subsequence of (x_n) whose limit is $\inf\{x_n \mid n \in \mathbb{N}\}$.

   (2) Let $(x_n), (y_n)$ be two bounded sequences of positive real numbers, then

   $$\limsup_{n\to\infty}(x_n y_n) \leq \limsup_{n\to\infty}(x_n) \limsup_{n\to\infty}(y_n)$$

   (3) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$, then there is a $\delta > 0$, such that f is continuous on $(a-\delta, a+\delta)$.

   (4) If $f : [0,1] \to \mathbb{R}$ is Riemann integrable, then the function $F(x) = \int_0^x f(t)dt$ exists and $F'(x) = f(x)$ for all $x \in (0,1)$.

   (5) If $f : \mathbb{R} \to \mathbb{R}$ is a smooth function, then for any $a \in \mathbb{R}$ and $x \in \mathbb{R}$, we have convergence

   $$\lim_{N\to\infty} \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(x).$$

2. True or False? If true, give a proof, if false, give a counter-example (5 points each)

   (a) Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Then

   $$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} (-1)^k f\left(\frac{k}{n}\right) = 0$$

   (b) Let $f : (0,\infty) \to (0,\infty)$ be uniformly continuous. Then

   $$\lim_{x\to\infty} \frac{f(x+1/x)}{f(x)} = 1.$$

3. Calculate and give your justifications.
   (a) (3 points)

   $$\lim_{x\to 0} \frac{a^x - 1}{x}, \quad a > 0$$

   (b) (3 points)

   $$\lim_{n\to\infty} \int_0^1 \frac{x^n}{1+x} dx$$

   (c) (4 points) For any integer $k \geq 0$, calculate

   $$\lim_{n\to\infty} \left(\frac{1^k + 3^k + \cdots (2n-1)^k}{n^{k+1}} \right)$$

4. (10 points) Define the sequences (a_n) and (b_n) as follows:

$$0 < b_1 < a_1, \quad a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}$$

   Show that (a_n) and (b_n) both converge to the same limit.

5. Find the radius of convergence of the following power series on $\mathbb{R}$

$$\sum_{n=0}^{\infty} (2 + (-1)^n)^n x^n$$

6. (10 points) Show the following series of function converge uniformly on $\mathbb{R}$

$$\sum_{n=1}^{\infty} n^2 x^2 e^{-n^2|x|}$$

7. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Prove that, for any $x_1, \cdots, x_n \in \mathbb{R}$, there exists x_0, such that

$$f(x_0) = \frac{1}{n}(f(x_1) + \cdots + f(x_n))$$

8. Let (X, d) be a metric space and $p \in X$ is a fixed point. For any $u \in X$, we define a function $f_u(x) = d(u, x) - d(p, x)$. Show that (5 points each)
   (a) For any $u \in X$, the function f_u is bounded.
   (b) For any $u, v \in X$, $d(u, v) = \sup_{x \in X} |f_u(x) - f_v(x)|$.

9. Show that (5 points each)
   (a) $x \geq \sin(x)$, for $x > 0$
   (b) $1 - x^2/2 \leq \cos(x)$, for $x > 0$ (you may use result from (a))

   Hint: Use mean value theorem.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $f(x)$ be differentiable at $x = a$. Let $\{x_n\}, \{y_n\}$ be two sequences converging to a, such that $x_n < a < y_n$. Prove that

$$f'(a) = \lim_{n \to \infty} \frac{f(y_n) - f(x_n)}{y_n - x_n}$$