Math 104 Midterm 1 Feb 18, 2021 (A)

Name: ____________________

- The midterm is 80 minutes. This is a closed book exam. You can take one cheatsheet with you. After the exam ends, you have 10 minutes to scan and upload your exam to gradescope.
- You can also use paper or iPad to write your exam. Latex is also OK.
- Please write the time slot that you take the exam on the first page of your submission.
- You can use theorems and results (in example and exercises) in the textbook, unless the problem requires you to prove from the definition.
- No calculator should be used. No discussion or searches on internet are allowed.
- If you have question during the exam, you may contact me using zoom direct message or via email.

Good Luck!

Question	Points	Score
1	20	
2	20	
3	10	
4	10	
5	10	
6	15	
7	15	
Total	100	

1. (20 points, 4 points each) True or False? Please provide your reasoning (no rigorous proof is needed).

   (1) Let $S \subset \mathbb{R}$ be a bounded subset. If S consists of only irrational numbers, then $\sup(S)$ is a irrational number.

   (2) There are no rational roots for $x^8 - x^4 + x - 1 = 0$.

   (3) If a convergent sequence (s_n) takes value 0 infinitely many times, then (s_n) converges to 0.

   (4) Let (s_n) be a sequence that enumerate all positive rational numbers of the form $n/3^k$, where n and k are positive integers. Then for any real number $t \geq 0$, there exists a subsequence of (s_n) that converges to t.

   (5) Let (s_n) be a sequence such that $\limsup(s_n) = 0$, then there exists an $N > 0$, such that for all $n > N$, $s_n \leq 0$.

2. (20 points, 5 points each) Compute the limits of the following sequences. Please provide intermediate steps and justifications.

   (1) $\frac{7n+3}{3n+7}$

   (2) $\frac{2^n+3^n}{3^n-2^n}$

   (3) $(1/2^n + 1/3^n)^{1/n}$

   (4) $(n^2+n)^{1/n}$

   Useful formula (1) for any positive number a, $\lim a^{1/n} = 1$. (2) $\lim n^{1/n} = 1$.

3. (10 points) Let s_n and t_n be Cauchy sequences in $\mathbb{R}$. Prove that $s_n + t_n$ is also a Cauchy sequence. You can only use the definitions to prove this statement.

4. (10 points) Let a_n, b_n be bounded sequences of real numbers. Prove that there exists a real number $t \in \mathbb{R}$, such that for any $\epsilon > 0$, the set $\{n \in \mathbb{N} \mid |a_n - b_n - t| < \epsilon\}$ is infinite.

5. (10 points) Let $s_n = (-1)^n/(1 + (1/n))$. Find all subsequential limits of s_n. Please justify your results, i.e., give subsequences that converges to these subsequential limits, and show that there are no other possible subsequential limits.

6. (15 point) Let s_n be a sequence of positive numbers that converges to 0. Show that there exists a subsequence $(s_{n_k})_k$ of s_n satisfying the condition that $s_{n_{k+1}} \leq \frac{1}{2} s_{n_k}$ for all $k \geq 1$.

7. (15 point) For any real number x, let $\lfloor x \rfloor$ denote the largest integer $n \leq x$. For any $t \in \mathbb{R}$, show that the following sequence $t_n = 2^{-n} \sum_{j=0}^{n} \lfloor 2^j t \rfloor$ is convergent, and find its limit.

   Hint: You may use the result that $\lim n^k/a^n = 0$ for all real number $a > 1$, and real number $k \geq 0$. And you can use the formula $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$. Also note that $0 \leq x - \lfloor x \rfloor < 1$.