There are 2 versions, with minor differences. I will do version-A first. then version B in purple.

1. (20 points, 4 points each) True or False? Please provide your reasoning (no rigorous proof is needed).

(1) Let $S \subset \mathbb{R}$ be a bounded subset. If S consists of only irrational numbers, then $\sup(S)$ is a irrational number.

(2) There are no rational roots for $x^8 - x^4 + x - 1 = 0$.

(3) If a convergent sequence (s_n) takes value 0 infinitely many times, then (s_n) converges to 0.

(4) Let (s_n) be a sequence that enumerate all positive rational numbers of the form $n/3^k$, where n and k are positive integers. Then for any real number $t \geq 0$, there exists a subsequence of (s_n) that converges to t.

(5) Let (s_n) be a sequence such that $\limsup(s_n) = 0$, then there exists an $N > 0$, such that for all $n > N$, $s_n \leq 0$.

1. (1) False. For example, let $S = \{-\frac{\sqrt{2}}{n} \mid n \in \mathbb{N}\}$.

then $\sup(S) = 0$.

Version B: False. Let $S = \{x \in \mathbb{Q} \mid x < \sqrt{2}\}$. $\sup(S) = \sqrt{2}$.

(2) False. The set of rational roots for such monic polynomial are integers., and it divides the constant term (−1 here). Hence, suffice to check if +1, −1 are roots. It turns out $x = 1$ is root:

(3) True. (S_n) takes value 0 infinitely many times means the set $A = \{n \in \mathbb{N} \mid S_n = 0\}$ is infinite. (It doesn't mean (S_n) is a constant seq.) Since one assumes (S_n) is convergent, hence all subsequence has the same limit, say α. In particular, the sub seq $\{S_n\}_{n \in A}$, the constant zero subseq converge to α, hence $\alpha = 0$.

If we don't assume S_n is convergent, the statement is false.

e.g. $S_n = \begin{cases} 0 & n \text{ even} \\ n & n \text{ odd.} \end{cases}$

True.

(4) Let $A = \{ r \mid r \in \mathbb{Q},$ and there exists $n, k \in \mathbb{N}$, such that $r = \frac{n}{3^k} \}$.

The sequence (S_n) enumerates the set A, i.e., every element in A appears in (S_n) once and exactly once.

Now, for any real $t \geq 0$, one need to show that $\{ n \mid |S_n - t| < \varepsilon \}$ is infinite., this is equivalent of showing $\{ a \in A \mid |a - t| < \varepsilon \}$ is infinite. Let M be a large enough positive integer, such that $3^M \cdot \varepsilon > 1$, then for each $k > M$, we claim there exist an $m_k \in \mathbb{N}$, s.t. $3 \nmid m_k$, and $\left| \frac{m_k}{3^k} - t \right| \leq \frac{1}{3^k}$. Indeed, consider $3^k \cdot t \in \mathbb{R}_{\geq 0}$, it lies in an interval $[\alpha, \alpha+1]$, with $\alpha \in \mathbb{Z}$, $\alpha \geq 0$. Then at least one of the end points, α or $\alpha+1$, is not divisible by 3. Let m_k equal to such an endpoint. Then,

$|m_k - 3^k t| \leq 1$, hence $\left| \frac{m_k}{3^k} - t \right| \leq \frac{1}{3^k}$. this proves the claim. Thus, the set $\{ \frac{m_k}{3^k} \mid k > M \text{ integer} \}$ is an infinite set, satisfying $\left| \frac{m_k}{3^k} - t \right| < \varepsilon$.

In exam, as long as you make the claim that, $\forall \varepsilon > 0$, there are infinitely many distinct rational numbers of the form $\frac{n}{3^k}$ inside $(t-\varepsilon, t+\varepsilon)$, you get point.

(5) False. For example, $S_n = \frac{1}{n}$, $\limsup S_n = 0$.

See Ross §12.

2. (20 points, 5 points each) Compute the limits of the following sequences. Please provide intermediate steps and justifications.

(1) $\frac{7n+3}{3n+7}$

(2) $\frac{2^n+3^n}{3^n-2^n}$

(3) $(1/2^n + 1/3^n)^{1/n}$

(4) $(n^2+n)^{1/n}$

Useful formula (1) for any positive number a, $\lim a^{1/n} = 1$. (2) $\lim n^{1/n} = 1$.

(1) $\lim \frac{7n+3}{3n+7} = \lim \frac{7+\frac{3}{n}}{3+\frac{7}{n}} = \frac{\lim(7+\frac{3}{n})}{\lim(3+\frac{7}{n})} = \frac{7}{3}$.

(2). $\lim \frac{2^n+3^n}{3^n-2^n} = \lim \frac{1+(\frac{2}{3})^n}{1-(\frac{2}{3})^n} = \frac{\lim(1+(\frac{2}{3})^n)}{\lim(1-(\frac{2}{3})^n)} = \frac{1}{1} = 1.$

(3) $\lim \left(\frac{1}{2^n}+\frac{1}{3^n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \left[\frac{1}{2^n}\cdot\left(1+\left(\frac{2}{3}\right)^n\right)\right]^{\frac{1}{n}} = \frac{1}{2} \lim \left(1+\left(\frac{2}{3}\right)^n\right)^{\frac{1}{n}}$

Since for all $n \in \mathbb{N}$, $1 \leq 1+(\frac{2}{3})^n \leq 2$, and

$\lim a^{\frac{1}{n}} = 1$ $\forall a > 0$, we have

$1 = \lim 1^{\frac{1}{n}} \leq \lim (1+(\frac{2}{3})^n)^{\frac{1}{n}} \leq \lim 2^{\frac{1}{n}} = 1$

Hence $\lim (1+(\frac{2}{3})^n)^{\frac{1}{n}} = 1$, and the original limit $= \frac{1}{2}$.

(4) $\lim (n^2+n)^{\frac{1}{n}} = \lim [n^2(1+\frac{1}{n})]^{\frac{1}{n}}$

$= \lim.\ n^{\frac{1}{n}} \cdot n^{\frac{1}{n}} \cdot (1+\frac{1}{n})^{\frac{1}{n}}$

since $\lim n^{\frac{1}{n}} = 1$, $\lim (1+\frac{1}{n})^{\frac{1}{n}} = 1$ see (3) for the argument.

we get $\lim (n^2+n)^{\frac{1}{n}} = \lim n^{\frac{1}{n}} \cdot \lim n^{\frac{1}{n}} \cdot \lim(1+\frac{1}{n})^{\frac{1}{n}}$

$= 1\cdot 1\cdot 1 = 1.$

#3 Prove that, if s_n and t_n are Cauchy, then $s_n + t_n$ is Cauchy.

Pf: We need to show that, $\forall \varepsilon > 0$, $\exists N$, s.t.

$$|s_{n_1} + t_{n_1} - (s_{n_2} + t_{n_2})| < \varepsilon \qquad \forall n_1, n_2 > N.$$

Since s_n is Cauchy, hence for any $\varepsilon_1 > 0$ $\exists N_1 > 0$, s.t.

$$|s_{n_1} - s_{n_2}| < \varepsilon_1 \qquad \forall n_1, n_2 > N_1$$

Similarly, since t_n is Cauchy, $\Rightarrow \forall \varepsilon_2 > 0$, $\exists N_2 > 0$. s.t.

$$|t_{n_1} - t_{n_2}| < \varepsilon_2 \qquad \forall n_1, n_2 > N_2.$$

Now, given $\varepsilon > 0$, choose $\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2}$, obtain the corresponding N_1, N_2, and set $N = \max\{N_1, N_2\}$. Indeed, $\forall n_1, n_2 > N$

$$|s_{n_1} + t_{n_1} - (s_{n_2} + t_{n_2})| = |(s_{n_1} - s_{n_2}) + (t_{n_1} - t_{n_2})|$$

$$\leq |s_{n_1} - s_{n_2}| + |t_{n_1} - t_{n_2}| \leq \varepsilon_1 + \varepsilon_2 = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

triangle inequality. $\quad \because n_1, n_2 \geq N \geq N_1$ and $n_1, n_2 \geq N \geq N_2$

Hence, such choice of N satisfies the requirement. #.

#4 Let a_n, b_n be bounded sequence of real numbers, show that, $\exists t \in \mathbb{R}$, s.t. $\forall \varepsilon > 0$, the set

$$\{n \mid |a_n - b_n - t| < \varepsilon\} \text{ is infinite.}$$

Pf: Suffice to show that, there is a subsequence of $(a_n - b_n)$ that converges, then we can choose t to be the limit of the subsequence. Since $(a_n - b_n)$ is a bounded sequence, by Bolzano-Weierstrass theorem,

$(a_n - b_n)$ has convergent subseq. #.

#5. Let $S_n = (-1)^n \frac{1}{1+\frac{1}{n}}$. Find all subsequential limits of (S_n).

Pf: We claim that ± 1 are all the subseq limits.

Indeed, $(S_n)_{n \text{ even}}$ converges to $+1$, and $(S_n)_{n \text{ odd}}$ converges to -1. Suppose $t \neq \pm 1$ is another subseq limit, then., $\exists\, \varepsilon > 0$, s.t. $(t-2\varepsilon, t+2\varepsilon)$ does not contain $+1, -1$.

In particular, $(-1-\varepsilon, -1+\varepsilon) \cap (t-\varepsilon, t+\varepsilon) = \emptyset$

$$(+1-\varepsilon, +1+\varepsilon) \cap (t-\varepsilon, t+\varepsilon) = \emptyset.$$

Since (S_n) can be partitioned into two subsequences, that converges to (-1) and $(+1)$ respectively. Hence, $\exists\, N > 0$, s.t. $\forall n > N$.

$$S_n \in (-1-\varepsilon, -1+\varepsilon) \cup (1-\varepsilon, 1+\varepsilon).$$

$$\Rightarrow \quad S_n \notin (t-\varepsilon, t+\varepsilon).$$

This shows, no subsequence can converge to t. Hence, if $t \neq \pm 1$, then t is not a subseq limit of (S_n).

#6 Suppose S_n is a seq of positive numbers that converge to 0, show that there exist a subseq satisfying the condition, that

$$S_{n_{k+1}} \leq \frac{1}{2} S_{n_k} \qquad \forall\, k \in \mathbb{N}.$$

Pf: Let $n_1 = 1$. Assume $n_1 < n_2 < \cdots < n_k$ are constructed, and satisfies the desired condition, we just need to show $\exists\, n_{k+1} > n_k$, s.t. $S_{n_{k+1}} \leq \frac{1}{2} S_{n_k}$. Indeed, by setting $\varepsilon = \frac{1}{2} S_{n_k}$, we get an $N > 0$, s.t $\forall n > N$,

$$|S_n - 0| < \varepsilon = \frac{1}{2} S_{n_k}.$$

Since S_n is positive, we get $0 < S_n < \frac{1}{2} S_{n_k}$ $\forall n > N$.

Clearly, $N \geq n_k$, otherwise $n_k > N$, but $S_{n_k} > \frac{1}{2} S_{n_k}$.

Hence, we may take $n_{k+1} = N+1.$, and get $S_{n_{k+1}} < \frac{1}{2} S_{n_k}$.

<u>#6 (version B)</u>: Let (S_n) be a seq of positive numbers that are not bounded above. Show that there is a subseq of (S_n), s.t. $S_{n_{k+1}} \geq 2 \cdot S_{n_k}$ $\forall k$.

<u>Pf</u>: Let $n_1 = 1$. Assume $n_1 < n_2 < \cdots < n_k$ are constructed, and satisfies $S_{n_{i+1}} > S_{n_i}$ for $i = 1, \cdots, k-1$, we now construct n_{k+1}, s.t, $n_{k+1} > n_k$ and $S_{n_{k+1}} \geq 2 \cdot S_{n_k}$. Indeed, since (S_n) is unbounded, hence $\forall M > 0$, the set $\{n \mid S_n > M\}$ is infinite. Hence, the set

$\{n \mid S_n > M\} \cap \{n \mid n > n_k\}$ is also infinite. Pick $M = 2 n_k$, and pick n_{k+1} be any element in

$\{n \mid S_n > 2n_k\} \cap \{n \mid n > n_k\}$, then n_{k+1} satisfies the requirement.

<u>#7</u> For any $t \in \mathbb{R}$, show that

$$t_n = 2^{-n} \sum_{j=0}^{n-1} \lfloor 2^j t \rfloor$$

converge.

Pf: $\because \quad 0 \le x - \lfloor x \rfloor < 1 \quad \therefore \quad x - 1 < \lfloor x \rfloor \le x$

Let $\quad a_n = 2^{-n} \sum_{j=0}^{n-1} (2^j t - 1)$

$\qquad\quad b_n = 2^{-n} \sum_{j=0}^{n-1} 2^j t.$

Then $\quad a_n \le t_n \le b_n$

Then
$$b_n = t \cdot 2^{-n} (1 + 2 + \cdots + 2^{n-1})$$
$$= t \cdot 2^{-n} (2^n - 1)$$
$$= t (1 - 2^{-n})$$

and
$$a_n = b_n - 2^{-n} \cdot n.$$

Since $\quad \lim b_n = \lim t(1 - 2^{-n}) = t$

$\qquad\quad \lim a_n = \lim b_n - \lim 2^{-n} \cdot n = t - 0 = t$

Thus,
$$t = \lim a_n \le \liminf t_n \le \limsup t_n \le \lim b_n = t$$

and we get $\quad \liminf t_n = \limsup t_n = t \Rightarrow \lim t_n = t.$

In version B, we had $\quad t_n = 2^{-n} \sum_{j=0}^{n} \lfloor 2^j t \rfloor$,

and the limit is $2t$.