Chapter 3

Sequences

Mathematicians have tried in vain to this day to discover some order in the sequence of prime numbers, and we have reason to believe that it is a mystery into which the human mind will never penetrate.

Leonhard Euler (1707–1783)

- A *sequence* is a function whose domain is the set $\mathbb{N}$ of natural numbers.
- A sequence $\{x_n\}$ is said to *converge* to a real number x, provided that for each $\varepsilon > 0$ there exists an integer N such that $n \geq N$ implies that $|x_n - x| < \varepsilon$.
  In this case we also say that $\{x_n\}$ converges to x, or x is the limit of $\{x_n\}$, and we write $x_n \to x$, or $\lim\limits_{n \Rightarrow \infty} x_n = x$. If $\{x_n\}$ does not converge, it is said to *diverge*.
- A sequence $\{x_n\}$ is said to be *bounded* if the range $\{x_n : \; n \in \mathbb{N}\}$ is a bounded set, that is, if there exists $M \geq 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.
- *Bolzano–Weierstrass Theorem*: Every bounded sequence has a convergent subsequence.
- Let $\{x_n\}_{n=1}^{\infty}$ be a sequence and for each $n \in \mathbb{N}$, set

$$y_n = \sup\{x_k : \; k \geq n\}.$$

  The *limit superior* of $\{x_n\}$, denoted by $\limsup\{x_n\}$ or $\overline{\lim}\{x_n\}$, is defined by

$$\overline{\lim}\{x_n\} = \inf\{y_n : \; n \in \mathbb{N}\} = \inf\{x : \; x = \sup\{x_k : \; k \geq n\} \text{ for some } n \in \mathbb{N}\}$$

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provided that the quantity on the right exists. Likewise we define the *limit inferior* by

$$\underline{\lim}\{x_n\} = \sup\{x : x = \inf\{x_k : \ k \geq n\} \text{ for some } n \in \mathbb{N}\}.$$

It is well known that if $\{x_n\}$ is a sequence, then $\{x_n\}$ has a limit if and only if the limit superior and the limit inferior exist and are equal.

- A sequence $\{x_n\}$ of real numbers is said to be a *Cauchy sequence* if for every $\varepsilon > 0$, there is an integer N such that
$$|x_n - x_m| < \varepsilon \text{ if } n \geq N \text{ and } m \geq N.$$
- Let $\{x_n\}_{n=1}^{\infty}$ be a sequence and let $\{n_k\}_{k=1}^{\infty}$ be any sequence of natural numbers such that $n_1 < n_2 < n_3 < \ldots$. The sequence $\{x_{n_k}\}_{k=1}^{\infty}$ is called a *subsequence* of $\{x_n\}_{n=1}^{\infty}$.

Problem 3.1 Show that each bounded sequence of real numbers has a convergent subsequence.

Problem 3.2 Show that if $\{x_n\}$ converges to l, then $\{|x_n|\}$ converges to $|l|$. What about the converse?

Problem 3.3 Let C be a real number such that $|C| < 1$. Show that $\lim_{n\to\infty} C^n = 0$.

Problem 3.4 Let $\{x_n\}$ be a sequence such that $\{x_{2n}\}$, $\{x_{2n+1}\}$, and $\{x_{3n}\}$ are convergent. Show that $\{x_n\}$ is convergent.

Problem 3.5 Let S be a nonempty subset of $\mathbb{R}$ which is bounded above. Set $s = \sup S$. Show that there exists a sequence $\{x_n\}$ in S which converges to s.

Problem 3.6 Let $\{x_n\}$ and $\{y_n\}$ be two real sequences such that

(a) $x_n \leq y_n$ for all n;

(b) $\{x_n\}$ is increasing;

(c) $\{y_n\}$ is decreasing.

Show that $\{x_n\}$ and $\{y_n\}$ are convergent and

$$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n \, .$$

When do we have equality of the limits?

Problem 3.7 Show that $\{x_n\}$ defined by

$$x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

is divergent.

Problem 3.8 Show that $\{x_n\}$ defined by

$$x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln(n)$$

is convergent.

Problem 3.9 Show that the sequence $\{x_n\}$ defined by

$$x_n = \int_1^n \frac{\cos(t)}{t^2} dt$$

is Cauchy.

Problem 3.10 Let $\{x_n\}$ be a sequence such that there exist $A > 0$ and $C \in (0, 1)$ for which

$$|x_{n+1} - x_n| \leq AC^n$$

for any $n \geq 1$. Show that $\{x_n\}$ is Cauchy. Is this conclusion still valid if we assume only

$$\lim_{n\to\infty} |x_{n+1} - x_n| = 0?$$

Problem 3.11 Show that if a subsequence $\{x_{n_k}\}$ of a Cauchy sequence $\{x_n\}$ is convergent, then $\{x_n\}$ is convergent.

Problem 3.12 Discuss the convergence or divergence of

$$x_n = \frac{n^2}{\sqrt{n^6+1}} + \frac{n^2}{\sqrt{n^6+2}} + \cdots + \frac{n^2}{\sqrt{n^6+n}} .$$

Problem 3.13 Discuss the convergence or divergence of

$$x_n = \frac{[\alpha] + [2\alpha] + \cdots + [n\alpha]}{n^2} ,$$

where $[x]$ denotes the greatest integer less than or equal to the real number x, and α is an arbitrary real number.

Problem 3.14 Discuss the convergence or divergence of

$$x_n = \frac{\alpha^n - \beta^n}{\alpha^n + \beta^n}$$

where α and β are real numbers such that $|\alpha| \neq |\beta|$.

Problem 3.15 (**Cesaro Average**) Let $\{x_n\}$ be a real sequence which converges to l. Show that the sequence

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

also converges to l. What about the converse? As an application of this, show that if $\{x_n\}$ is such that $\lim_{n\to\infty} x_{n+1} - x_n = l$, then

$$\lim_{n\to\infty} \frac{x_n}{n} = l \ .$$

Problem 3.16 Let $\{x_n\}$ be a real sequence with $x_n \neq 0$. Assume that

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = l \ .$$

Show that

(a) if $|l| < 1$, then $\lim_{n\to\infty} x_n = 0$;

(b) and if $|l| > 1$, then $\{x_n\}$ is divergent.

What happens when $|l| = 1$? As an application decide on convergence or divergence of

$$x_n = \frac{\alpha^n}{n^k} \ \text{ and } \ y_n = \frac{\alpha^n}{n!} \ .$$

Problem 3.17 Given $x \geq 1$, show that

$$\lim_{n\to\infty} \left(2\sqrt[n]{x} - 1\right)^n = x^2 \ .$$

Problem 3.18 Show that

$$\lim_{n\to\infty} \frac{\left(2\sqrt[n]{n} - 1\right)^n}{n^2} = 1 \ .$$

Problem 3.19 Let $\{x_n\}$ defined by

$$x_1 = 1 \text{ and } x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

Show that $\{x_n\}$ is convergent and find its limit.

Problem 3.20 Let $\{x_n\}$ be a sequence defined by

$$x_1 = 1\,, \text{ and } x_{n+1} = \sqrt{x_n^2 + \frac{1}{2^n}}\,.$$

Show that $\{x_n\}$ is convergent.

Problem 3.21 For any $n \in \mathbb{N}$ set $I_n = \displaystyle\int_0^{\pi/2} \cos^n(t)dt$, known as Wallis integrals .

1. Show that $(n+2)I_{n+2} = (n+1)I_n$. Then use it to find explicitly I_{2n} and I_{2n+1}.
2. Show that $\displaystyle\lim_{n\to\infty} \frac{I_{n+1}}{I_n} = 1$.
3. Show that $\{(n+1)I_nI_{n+1}\}$ is a constant sequence. Then conclude that

$$\lim_{n\to\infty} I_n\sqrt{2n} = \sqrt{\pi}.$$

Problem 3.22 Consider the sequence

$$x_n = \frac{n!}{\sqrt{n}}\left(\frac{e}{n}\right)^n,\ n = 1, \dots .$$

1. Show that $\{\ln(x_n)\}$ is convergent. Use this to show that $\{x_n\}$ is convergent.
2. Use Wallis integrals to find the limit of $\{x_n\}$.
3. Use 1. and 2. to prove the Stirling formula

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

   when $n \to \infty$.

Problem 3.23 Find the limit superior and limit inferior of the sequence $\{x_n\}$, where

- $x_n = 1 + (-1)^n + \frac{1}{2^n}$
- $x_n = 2^n$

Problem 3.24 Let $\{x_n\}$ be a bounded sequence. Prove there exists a subsequence of $\{x_n\}$ which converges to $\liminf\limits_{n\to\infty} x_n$. Show that the same conclusion holds for $\limsup\limits_{n\to\infty} x_n$.

Problem 3.25 Let $\{x_n\}$ be a sequence and let $\{x_{n_k}\}$ be any of its subsequences. Show that

$$\liminf_{n\to\infty} x_n \leq \liminf_{n_k\to\infty} x_{n_k} \leq \limsup_{n_k\to\infty} x_{n_k} \leq \liminf_{n\to\infty} x_n.$$

In particular, if $\{x_{n_k}\}$ is convergent, then

$$\liminf_{n\to\infty} x_n \leq \lim_{n_k\to\infty} x_{n_k} \leq \limsup_{n\to\infty} x_n.$$

Is the converse true? That is, for any l between $\liminf\limits_{n\to\infty} x_n$ and $\limsup\limits_{n\to\infty} x_n$, there exists a subsequence $\{x_{n_k}\}$ which converges to l.

Problem 3.26 If (x_n) and (y_n) are bounded real sequences, show that

$$\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.$$

Do we have equality?

Problem 3.27 If $x_n > 0$, $n = 1, 2, \ldots$, show that

$$\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} \leq \liminf_{n\to\infty} \sqrt[n]{x_n} \leq \limsup_{n\to\infty} \sqrt[n]{x_n} \leq \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

Deduce that if $\lim\limits_{n\to\infty} \dfrac{x_{n+1}}{x_n}$ exists, then $\lim_{n\to\infty} \sqrt[n]{x_n}$ exists. What happened to the converse?