## Solutions

**Solution 3.1**

Let $(x_n : n \in \mathbb{N})$ be a bounded sequence, say $|x_n| \leq M$ for all $n$.

Let $I_0 = [-M, M]$, $a_0 = -M$, and $b_0 = M$, so that $I_0 = [a_0, b_0]$ and $I_0$ contains infinitely many of the $x_n$ (in fact, all of them).

We construct inductively a sequence of intervals $I_k = [a_k, b_k]$ such that $I_k$ contains infinitely many of the $x_n$ and $b_k - a_k = 2M/2^k$. This certainly holds for $k = 0$.

Suppose it holds for some value of $k$. Then at least one of the intervals $[a_k, (a_k + b_k)/2]$ and $[(a_k + b_k)/2, b_k]$ contains infinitely many of the $x_n$. If the former, then let $a_{k+1} = a_k$, $b_{k+1} = (a_k + b_k)/2$. Otherwise, let $a_{k+1} = (a_k + b_k)/2$, $b_{k+1} = b_k$. In either case, the interval $I_{k+1} = [a_{k+1}, b_{k+1}]$ contains infinitely many of the $x_n$, and

$$b_{k+1} - a_{k+1} = \frac{1}{2}(b_k - a_k) = \left(\frac{1}{2}\right)^{k+1} \times 2M.$$

This completes the inductive construction.

Clearly $a_0 \leq a_1 \leq a_2 \leq \cdots \leq b_2 \leq b_1 \leq b_0$. Thus $(a_n)$ is an increasing bounded sequence, so by completeness has a limit, say $x$. Moreover since each $b_k$ is an upper bound for $(a_n)$ and $x$ is the supremum, $x \leq b_k$ for each $k$. Thus $a_k \leq x \leq b_k$ for every $k$. In other words, $x \in I_k$ for every $k$.

We now construct inductively a subsequence $(x_{n_k})$ of $(x_n)$ such that $x_{n_k} \in I_k$ for every $k$. Let $x_{n_0} = x_0$. Assuming $x_{n_k}$ has been chosen, let $n_{k+1}$ be the least $n > n_k$ such that $x_n \in I_{k+1}$. Then $(x_{n_k})$ is a subsequence of $(x_n)$, and $x_{n_k} \in I_k$ for every $k$.

Since $x_{n_k}$ and $x$ both lie in the same interval $I_k$ of length $2M/2^k$, it follows that

$$|x_{n_k} - x| \leq \left(\frac{1}{2}\right)^{k} \times 2M$$

and so $|x_{n_k} - x| \to 0$ as $n \to \infty$. Thus $(x_{n_k})$ is a convergent subsequence of $(x_n)$, as required.

**Solution 3.2**

Note that for any real numbers $x, y \in \mathbb{R}$, we have

$$\Big||x| - |y|\Big| \leq |x - y| \, .$$

Since $\{x_n\}$ converges to $l$, then for any $\varepsilon > 0$, there exists $n_0 \geq 1$ such that for any $n \geq n_0$, we have

$$|x_n - l| < \varepsilon \, .$$

Hence

$$\Big||x_n| - |l|\Big| < \varepsilon$$

for any $n \geq n_0$. This obviously implies the desired conclusion. For the converse, take $x_n = (-1)^n$, for $n = 0, \dots$ . Then we have $|x_n| = 1$ which means that $\{|x_n|\}$ converges to 1. But $\{x_n\}$ does not converge. Note that if $l = 0$, then the converse is true.

**Solution 3.3**

If $C = 0$, then the conclusion is obvious. Assume first $0 < C < 1$. Then the sequence $\{C^n\}$ is decreasing and bounded below by 0. So it has a limit $L$. Let us prove that $L = 0$. We have $C^{n+1} = CC^n$ so by passing to the limit we get $L = CL$ which implies $L = 0$. If $-1 < -C < 0$, then we use $(-C)^n = (-1)^n C^n$ and the fact that the product of a bounded sequence with a sequence which converges to 0 also converges to 0 to get $\lim_{n\to\infty} (-C)^n = 0$. Therefore, for any $-1 < C < 1$, we have $\lim_{n\to\infty} C^n = 0$.

**Solution 3.4**

If $\{x_n\}$ is convergent, then all subsequences of $\{x_n\}$ are convergent and converge to the same limit. Therefore, let us show that the three subsequences converge to the same limit. Write

$$\lim_{n\to\infty} x_{2n} = \alpha_1, \ \lim_{n\to\infty} x_{2n+1} = \alpha_2, \text{ and } \lim_{n\to\infty} x_{3n} = \alpha_3 .$$

The sequence $\{x_{6n}\}$ is a subsequence of both sequences $\{x_{2n}\}$ and $\{x_{3n}\}$. Hence $\{x_{6n}\}$ converges and forces the following:

$$\lim_{n\to\infty} x_{6n} = \lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{3n}$$

or $\alpha_1 = \alpha_3$. On the other hand, the sequence $\{x_{6n+3}\}$ is a subsequence of both sequences $\{x_{2n+1}\}$ and $\{x_{3n}\}$. Hence $\{x_{6n+3}\}$ converges and forces the following:

$$\lim_{n\to\infty} x_{6n+3} = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} x_{3n}$$

or $\alpha_2 = \alpha_3$. Hence $\alpha_1 = \alpha_2 = \alpha_3$. Let us write

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n+1} = l$$

and let us prove that $\lim_{n\to\infty} x_n = l$. Let $\varepsilon > 0$. There exist $N_0 \geq 1$ and $N_1 \geq 1$ such that

$$\begin{cases} |x_{2n} - l| < \varepsilon & \text{for all } n \geq N_0, \\ |x_{2n+1} - l| < \varepsilon & \text{for all } n \geq N_1. \end{cases}$$

Set $N = \max\{2N_0, 2N_1 + 1\}$. Let $n \geq N$. If $n = 2k$, then we have $k \geq N_0$ since $n \geq N \geq 2N_0$. Using the above inequalities we get $|x_{2k} - l| < \varepsilon$ or $|x_n - l| < \varepsilon$. A similar argument when $n$ is odd will yield the same inequality. Therefore

$$|x_n - l| < \varepsilon$$

for any $n \geq N$. This completes the proof of our statement.

**Solution 3.5**

By the characterization of the supremum, we know that for any $\varepsilon > 0$ there exists $x \in S$ such that

$$s - \varepsilon < x \leq s \ .$$

So for any $n \geq 1$, there exists $x_n \in S$ such that

$$s - \frac{1}{n} < x_n \leq s \ .$$

Since $\left\{\frac{1}{n}\right\}$ goes to 0, given $\varepsilon > 0$, there exists $n_0 \geq 1$ such that for any $n \geq n_0$ we have $\frac{1}{n} < \varepsilon$. So for any $n \geq n_0$ we have

$$s - \varepsilon < s - \frac{1}{n} < x_n \leq s < s + \varepsilon \ ,$$

which implies

$$|x_n - s| < \varepsilon \ ,$$

which translates into $\lim_{n\to\infty} x_n = s$.

**Solution 3.6**

Since $\{y_n\}$ is decreasing, we have $y_n \leq y_1$ for $n \geq 1$. So for any $n \geq 1$ we have $x_n \leq y_n \leq y_1$. This implies that $\{x_n\}$ is bounded above. Since it is increasing it converges. Similar argument shows that $\{y_n\}$ is bounded below and therefore converges as well. From (a) we get the desired inequality on the limits. In order to have the equality of the limits we must have $\lim_{n\to\infty} y_n - x_n = 0$. This result is useful when dealing with nested intervals in $\mathbb{R}$ and alternating real series.

**Solution 3.7**

We have

$$x_{2n} - x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

for any $n \geq 1$. So

$$\frac{1}{n+n} + \frac{1}{n+n} + \cdots + \frac{1}{2n} \leq x_{2n} - x_n$$

or $\frac{1}{2} \leq x_{2n} - x_n$. This clearly implies that $\{x_n\}$ fails to be Cauchy. Therefore it diverges.

**Solution 3.8**

Though real functions will be handled in the next chapters, here we will use the integral definition of the logarithm function. In particular, we have

$$\ln(x) = \int_1^x \frac{1}{t} dt \ .$$

In this case if $0 < a < b$, then we have

$$\frac{b-a}{b} \leq \int_a^b \frac{1}{t} dt \leq \frac{b-a}{a}.$$

Since

$$\ln(n) = \int_1^n \frac{1}{t}dt = \sum_{k=1}^{n-1} \int_k^{k+1} \frac{1}{t}dt ,$$

we get

$$\ln(n) \le \sum_{k=1}^{n-1} \frac{k+1-k}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} .$$

Hence

$$x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \ln(n) + \frac{1}{n} > 0 .$$

On the other hand, we have

$$x_{n+1} - x_n = \frac{1}{n+1} - \ln(n+1) + \ln(n) = \frac{1}{n+1} - \int_n^{n+1} \frac{1}{t}dt < 0 .$$

These two inequalities imply that $\{x_n\}$ is decreasing and bounded below by 0. Therefore $\{x_n\}$ is convergent. Its limit is known as the Euler constant.

**Solution 3.9**

For any natural integers $n < m$ we have

$$\left|\int_n^m \frac{\cos(t)}{t^2}dt\right| \le \int_n^m \frac{|\cos(t)|}{t^2}dt \le \int_n^m \frac{1}{t^2}dt = \left[-\frac{1}{t}\right]_n^m = \frac{1}{m} - \frac{1}{n}.$$

Since $\lim_{n\to\infty} \frac{1}{n} = 0$, then for any $\varepsilon > 0$, there exists $n_0 \ge 1$ such that for any $n \ge n_0$ we have $\frac{1}{n} < \varepsilon$. So for $n, m \ge n_0$, $n \le m$, we have

$$|x_n - x_m| = \left|\int_n^m \frac{\cos(t)}{t^2}dt\right| \le \frac{1}{m} - \frac{1}{n} < \varepsilon ,$$

which shows that $\{x_n\}$ is a Cauchy sequence.

**Solution 3.10**

Let $n \ge 1$ and $h \ge 1$. We have

$$|x_{n+h} - x_n| = \left|\sum_{k=0}^{h-1} x_{n+k+1} - x_{n+k}\right| \le \sum_{k=0}^{h-1} |x_{n+k+1} - x_{n+k}| .$$

Our assumption on $\{x_n\}$ implies

$$|x_{n+h} - x_n| \le \sum_{k=0}^{h-1} AC^{n+k} = AC^n \frac{1-C^h}{1-C} < A\frac{C^n}{1-C} .$$

Since $0 < C < 1$, $\lim_{n\to\infty} C^n = 0$. Hence

$$\lim_{n\to\infty} A\frac{C^n}{1-C} = 0 .$$

This will force $\{x_n\}$ to be Cauchy. The second part of the statement is not true. Indeed, take $x_n = \sqrt{n}$. Then we have

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0 .$$

But the sequence $\{x_n\}$ is divergent.

**Solution 3.11**

Set $\lim\limits_{n_k\to\infty} x_{n_k} = L$. Let us show that $\{x_n\}$ converges to $L$. Let $\varepsilon > 0$. Since $\{x_n\}$ is Cauchy, there exists $n_0 \geq 1$ such that for any $n, m \geq n_0$ we have

$$|x_n - x_m| < \frac{\varepsilon}{2} .$$

Since $\lim\limits_{n_k\to\infty} x_{n_k} = L$, there exists $k_0 \geq 1$ such that for any $k \geq k_0$ we have

$$|x_{n_k} - L| < \frac{\varepsilon}{2} .$$

For $k$ big enough to have $n_k \geq n_0$ we get

$$|x_n - L| \leq |x_n - x_{n_k}| + |x_{n_k} - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for any $n \geq n_0$. This completes the proof.

**Solution 3.12**

Note that for any $k = 1, \ldots, n$, we have

$$\frac{n^2}{\sqrt{n^6+n}} \leq \frac{n^2}{\sqrt{n^6+k}} \leq \frac{n^2}{\sqrt{n^6}} = \frac{1}{n}$$

which implies

$$n\frac{n^2}{\sqrt{n^6+n}} \leq x_n \leq n\frac{1}{n}$$

or

$$\frac{n^3}{\sqrt{n^6+n}} \leq x_n \leq 1 .$$

Because

$$\frac{n^3}{\sqrt{n^6+n}} = \frac{n^3}{n^3\sqrt{1+\dfrac{1}{n^2}}} = \frac{1}{\sqrt{1+\dfrac{1}{n^2}}}$$

and $\lim\limits_{n\to\infty} \frac{1}{n^2} = 0$, then $\lim\limits_{n\to\infty} \frac{n^3}{\sqrt{n^6+n}} = 1$. The Squeeze Theorem forces the conclusion

$$\lim_{n\to\infty} \frac{n^2}{\sqrt{n^6+1}} + \frac{n^2}{\sqrt{n^6+2}} + \cdots + \frac{n^2}{\sqrt{n^6+n}} = 1 .$$

**Solution 3.13**

By definition of the greatest integer function $[\cdot]$, we have

$$[x] \leq x < [x] + 1$$

for any real number $x$. This will easily imply $x - 1 < [x] \leq x$. So

$$\frac{(\alpha-1)+(2\alpha-1)+\cdots+(n\alpha-1)}{n^2} < \frac{[\alpha]+[2\alpha]+\cdots+[n\alpha]}{n^2} \leq \frac{\alpha+2\alpha+\cdots+n\alpha}{n^2}$$

or

$$\frac{(1+2+\cdots+n)\alpha - n}{n^2} < \frac{[\alpha]+[2\alpha]+\cdots+[n\alpha]}{n^2} \leq \frac{(1+2+\cdots+n)\alpha}{n^2} .$$

The algebraic identity $1+2+\cdots+m = \dfrac{m(m+1)}{2}$ for any natural number $m \geq 1$ gives

$$\frac{\frac{n(n+1)}{2}\alpha - n}{n^2} < \frac{[\alpha]+[2\alpha]+\cdots+[n\alpha]}{n^2} \leq \frac{\frac{n(n+1)}{2}\alpha}{n^2}$$

or

$$\frac{(n+1)\alpha}{2n} - \frac{1}{n} < \frac{[\alpha]+[2\alpha]+\cdots+[n\alpha]}{n^2} \leq \frac{(n+1)\alpha}{2n} .$$

Since

$$\lim_{n\to\infty} \frac{(n+1)\alpha}{2n} - \frac{1}{n} = \frac{\alpha}{2} \quad \text{and} \quad \lim_{n\to\infty} \frac{(n+1)\alpha}{2n} = \frac{\alpha}{2} ,$$

the Squeeze Theorem implies $\lim\limits_{n\to\infty} x_n = \dfrac{\alpha}{2}$.

**Solution 3.14**

We have two cases, either $|\alpha| < |\beta|$ or $|\alpha| > |\beta|$. Assume first that $|\alpha| < |\beta|$. Set $r = \frac{\alpha}{\beta}$. Then algebraic manipulation gives

$$x_n = \frac{r^n - 1}{r^n + 1} .$$

Since $|r| < 1$, then $\lim\limits_{n\to\infty} r^n = 0$, and we have $\lim\limits_{n\to\infty} x_n = -1$. Finally, if $|\alpha| > |\beta|$, then we use

$$\frac{\alpha^n - \beta^n}{\alpha^n + \beta^n} = -\frac{\beta^n - \alpha^n}{\beta^n + \alpha^n}$$

and the same argument given before will imply

$$\lim_{n\to\infty} x_n = -\lim_{n\to\infty} \frac{\beta^n - \alpha^n}{\beta^n + \alpha^n} = 1 .$$

**Solution 3.15**

Let $\varepsilon > 0$. Since $\lim\limits_{n\to\infty} x_n = l$, there exists $N_0 \geq 1$ such that for any $n \geq N_0$ we have

$$|x_n - l| < \frac{\varepsilon}{2} .$$

On the other hand, we have

$$y_n - l = \frac{x_1 + x_2 + \cdots + x_n}{n} - l = \frac{(x_1 - l) + (x_2 - l) + \cdots + (x_n - l)}{n}$$

or

$$y_n - l = \frac{(x_1 - l) + (x_2 - l) + \cdots + (x_{N_0-1} - l)}{n} + \frac{(x_{N_0} - l) + \cdots + (x_n - l)}{n}$$

for any $n \geq N_0$. Since

$$\lim_{n\to\infty} \frac{(x_1 - l) + (x_2 - l) + \cdots + (x_{N_0-1} - l)}{n} = 0.$$

Then, there exists $N_1 \geq 1$ such that

$$\left| \frac{(x_1 - l) + (x_2 - l) + \cdots + (x_{N_0-1} - l)}{n} \right| < \frac{\varepsilon}{2}$$

for any $n \geq N_1$. Set $N \max\{N_0, N_1\}$, then for any $n \geq N$ we have

$$|y_n - l| \leq \left| \frac{(x_1 - l) + (x_2 - l) + \cdots + (x_{N_0-1} - l)}{n} \right| + \left| \frac{(x_{N_0} - l) + \cdots + (x_n - l)}{n} \right|$$

or

$$|y_n - l| \leq \left| \frac{(x_1 - l) + (x_2 - l) + \cdots + (x_{N_0-1} - l)}{n} \right| + \frac{|x_{N_0} - l| + \cdots + |x_n - l|}{n}$$

which implies

$$|y_n - l| < \frac{\varepsilon}{2} + \frac{n - N_0}{n}\frac{\varepsilon}{2} < \varepsilon\ .$$

This completes the proof of our statement. For the converse take $x_n = (-1)^n$. Then we have

$$y_n = \begin{cases} -\dfrac{1}{n} & \text{if } n \text{ is odd,} \\ \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Obviously this will imply that $\lim\limits_{n\to\infty} y_n = 0$ while $\{x_n\}$ is well known to be divergent. Finally, let $\{x_n\}$ be a sequence such that $\lim\limits_{n\to\infty} x_{n+1} - x_n = l$. Set

$$y_n = \frac{(x_2 - x_1) + (x_3 - x_2) + \cdots + (x_{n+1} - x_n)}{n}\ .$$

Then from the first part we have $\lim\limits_{n\to\infty} y_n = l$. But

$$y_n = \frac{x_{n+1} - x_1}{n}$$

which implies $x_{n+1} = ny_n + x_1$. Hence

$$\frac{x_n}{n} = \frac{n-1}{n} y_{n-1} + \frac{x_1}{n}.$$

Since

$$\lim_{n\to\infty} \frac{n-1}{n} = 1\ ,\ \lim_{n\to\infty} y_n = l\ ,\ \text{and}\ \lim_{n\to\infty} \frac{x_1}{n} = 0$$

we get

$$\lim_{n\to\infty} \frac{x_n}{n} = l \ .$$

**Solution 3.16**

Assume first that $|l| < 1$. Let $\varepsilon = \dfrac{1-|l|}{2}$. Then we have $\varepsilon > 0$. Since

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = l$$

we get

$$\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = |l| \ .$$

Thus there exists $N_0 \geq 1$ such that for any $n \geq N_0$

$$\left|\frac{|x_{n+1}|}{|x_n|} - |l|\right| < \varepsilon$$

which implies

$$|l| - \varepsilon < \frac{|x_{n+1}|}{|x_n|} < |l| + \varepsilon$$

for any $n \geq N_0$. By definition of $\varepsilon$ we get

$$\frac{|x_{n+1}|}{|x_n|} < \frac{|l|+1}{2} < 1 \ .$$

In particular, we have for any $n \geq N_0$

$$|x_{n+1}| < \left(\frac{|l|+1}{2}\right)^{n-N_0+1} |x_{N_0}|.$$

Since $\displaystyle\lim_{n\to\infty} \left(\frac{|l|+1}{2}\right)^{n-N_0+1} = 0$, we get $\displaystyle\lim_{n\to\infty} |x_n| = 0$ which obviously implies $\displaystyle\lim_{n\to\infty} x_n = 0$. This completes the proof of the first part. Now assume $|l| > 1$. Since again

$$\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = |l| \ ,$$

the same proof as above gives the existence of $N_0 \geq 1$ such that

$$\left(\frac{|l|+1}{2}\right)^{n-N_0+1} |x_{N_0}| < |x_{n+1}|$$

for any $n \geq N_0$. And since $\displaystyle\lim_{n\to\infty} \left(\frac{|l|+1}{2}\right)^{n-N_0+1} = \infty$, we get $\displaystyle\lim_{n\to\infty} |x_n| = \infty$. Hence the sequence $\{x_n\}$ is not bounded and therefore is divergent. Finally if we assume $|l| = 1$, then it is possible that $\{x_n\}$ may be convergent or divergent. For example, take $x_n = n^\alpha$, then we have $l = 1$. But the sequence only converges if $\alpha \leq 0$, otherwise it diverges. For the sequences

$$x_n = \frac{\alpha^n}{n^k} \ \text{and} \ y_n = \frac{\alpha^n}{n!} \ ,$$

we have

$$\frac{x_{n+1}}{x_n} = \alpha\left(\frac{n}{n+1}\right)^k \text{ and } \frac{y_{n+1}}{y_n} = \alpha\frac{n!}{(n+1)!} = \alpha\frac{1}{n+1}\ .$$

Hence

$$\lim_{n\to\infty}\frac{x_{n+1}}{x_n} = \alpha \text{ and } \lim_{n\to\infty}\frac{y_{n+1}}{y_n} = 0\ .$$

In particular, we have

$$\begin{cases} \lim\limits_{n\to\infty} x_n = 0 & \text{if } |\alpha| < 1, \\ \{x_n\} \text{ is divergent} & \text{if } |\alpha| > 1. \end{cases}$$

And if $|\alpha| = 1$, then the sequence in question is $\left\{\frac{1}{n^k}\right\}$ or $\left\{\frac{(-1)^n}{n^k}\right\}$ which is easy to conclude. For the sequence $\{y_n\}$ we have $\lim\limits_{n\to\infty} y_n = 0$ regardless of the value of $\alpha$.

**Solution 3.17**

Without loss of generality, we may assume $1 < x$. First note that

$$0 < \left(\sqrt[n]{x} - 1\right)^2 = \sqrt[n]{x^2} - 2\sqrt[n]{x} + 1\ ,$$

which implies $2\sqrt[n]{x} - 1 < \sqrt[n]{x^2}$. Hence

$$\left(2\sqrt[n]{x} - 1\right)^n < \left(\sqrt[n]{x^2}\right)^n = x^2\ .$$

On the other hand, we have

$$\left(2\sqrt[n]{x} - 1\right)^n = x^2\left(\frac{2\sqrt[n]{x} - 1}{\sqrt[n]{x^2}}\right)^n = x^2\left(\frac{2}{\sqrt[n]{x}} - \frac{1}{\sqrt[n]{x^2}}\right)^n = x^2\left(1 - \left(1 - \frac{1}{\sqrt[n]{x}}\right)^2\right)^n\ .$$

Since $(1-h)^n \geq 1 - nh$, for any $h \geq 0$ and $n \geq 1$ we get

$$\left(1 - \left(1 - \frac{1}{\sqrt[n]{x}}\right)^2\right)^n \geq 1 - n\left(1 - \frac{1}{\sqrt[n]{x}}\right)^2\ ,$$

and

$$x = \left(\sqrt[n]{x} - 1 + 1\right)^n \geq 1 + n\left(\sqrt[n]{x} - 1\right) > n\left(\sqrt[n]{x} - 1\right)\ ,$$

which implies

$$\left(\sqrt[n]{x} - 1\right)^2 < \frac{x^2}{n^2}\ .$$

Hence

$$\left(2\sqrt[n]{x} - 1\right)^n \geq x^2\left(1 - n\left(1 - \frac{1}{\sqrt[n]{x}}\right)^2\right) = x^2\left(1 - n\frac{(\sqrt[n]{x} - 1)^2}{\sqrt[n]{x^2}}\right)\ ,$$

or

$$\left(2\sqrt[n]{x} - 1\right)^n > x^2\left(1 - \frac{x^2}{n\sqrt[n]{x^2}}\right)\ .$$

Putting all the inequalities together we get

$$x^2\left(1-\frac{x^2}{n\sqrt[n]{x^2}}\right) < \left(2\sqrt[n]{x}-1\right)^n < x^2\ .$$

The Squeeze Theorem will then imply

$$\lim_{n\to\infty}\left(2\sqrt[n]{x}-1\right)^n = x^2\ ,$$

since

$$\lim_{n\to\infty} x^2\left(1-\frac{x^2}{n\sqrt[n]{x^2}}\right) = x^2\ .$$

**Solution 3.18**

In the previous problem we showed

$$x^2\left(1-n\frac{\left(\sqrt[n]{x}-1\right)^2}{\sqrt[n]{x^2}}\right) < \left(2\sqrt[n]{x}-1\right)^n < x^2\ ,$$

for any $x > 1$ and $n \geq 1$. Take $x = n$, we get

$$n^2\left(1-n\frac{\left(\sqrt[n]{n}-1\right)^2}{\sqrt[n]{n^2}}\right) \leq \left(2\sqrt[n]{n}-1\right)^n \leq n^2\ ,$$

which implies

$$1-n\frac{\left(\sqrt[n]{n}-1\right)^2}{\sqrt[n]{n^2}} \leq \frac{\left(2\sqrt[n]{n}-1\right)^n}{n^2} \leq 1\ .$$

In order to complete the proof of our statement we only need to show

$$\lim_{n\to\infty} n\frac{\left(\sqrt[n]{n}-1\right)^2}{\sqrt[n]{n^2}} = 0\ .$$

Note that for $x \in [0,1]$ we have $0 \leq e^x - 1 \leq 3x$. Hence

$$0 \leq \sqrt[n]{n}-1 = e^{\frac{\ln(n)}{n}} - 1 \leq 3\frac{\ln(n)}{n}\ ,$$

because $\ln(n) \leq n$ for $n \geq 1$. So

$$0 \leq n\left(\sqrt[n]{n}-1\right)^2 \leq n9\frac{\ln(n)^2}{n^2} = 9\frac{\ln^2(n)}{n}\ .$$

Since $\lim\limits_{n\to\infty}\dfrac{\ln^2(n)}{n} = 0$, we conclude that

$$\lim_{n\to\infty} n\left(\sqrt[n]{n}-1\right)^2 = 0\ ,$$

which yields

$$\lim_{n\to\infty} n\frac{\left(\sqrt[n]{n}-1\right)^2}{\sqrt[n]{n^2}} = 0 \ .$$

**Solution 3.19**

Let us first show by induction that $0 \le x_n$ and $1 \le x_n^2 \le 2$. Obviously we have $0 \le 1$ and $1 \le 1^2 \le 2$. Assume that $0 \le x_n$ and $1 \le x_n^2 \le 2$. Then by the definition of $x_{n+1}$ we obtain easily $0 \le x_{n+1}$. On the other hand, we have

$$x_{n+1}^2 = \frac{1}{4}\left(x_n^2 + 4 + \frac{4}{x_n^2}\right) = \frac{1}{4}\left(x_n^2 + \frac{4}{x_n^2}\right) + 1 \ .$$

Since $(2-x_n)^2 = 4 - 4x_n^2 + x_n^4 \ge 0$ we get $\dfrac{x_n^4+4}{4x_n^2} \le 1$ or $\dfrac{1}{4}\left(x_n^2 + \dfrac{4}{x_n^2}\right) \le 1$. This will imply $x_{n+1}^2 \le 1+1 = 2$. So the induction argument gives the desired conclusion that is $x_n \ge 0$ and $1 \le x_n^2 \le 2$, for any $n \ge 1$. On the other hand, algebraic manipulations give

$$x_{n+1} - x_n = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) - x_n = \frac{2-x_n^2}{2x_n}$$

which implies $x_{n+1} - x_n \ge 0$ for any $n \ge 1$. Hence $\{x_n\}$ is an increasing bounded sequence. So it converges. Set $\lim\limits_{n\to\infty} x_n = l$. Then we have $l \ge 0$ and $1 \le l^2 \le 2$. Since $\{x_{n+1}\}$ also converges to $l$, we get

$$l = \frac{1}{2}\left(l + \frac{2}{l}\right) = \frac{l^2+2}{2l} \ ,$$

or $2l^2 = l^2 + 2$, which gives $l^2 = 2$ or $l = \sqrt{2}$. Note that the sequence $\{x_n\}$ is formed of rational numbers and its limit is irrational. One may generalize this problem to the sequence

$$x_1 = 1 \ \text{ and } \ x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right)$$

and show that $\{x_n\}$ converges to $\sqrt{\alpha}$ provided $\alpha \ge 0$.

**Solution 3.20**

Obviously the sequence $\{x_n\}$ is positive and since $x_{n+1} = \sqrt{x_n^2 + \frac{1}{2^n}} \ge \sqrt{x_n^2} = x_n$ in other words, the sequence $\{x_n\}$ is increasing. So in particular we have $x_n \ge x_1 = 1$ for any $n \ge 1$. Since

$$x_{n+1} - x_n = \sqrt{x_n^2 + \frac{1}{2^n}} - x_n = \frac{\frac{1}{2^n}}{\sqrt{x_n^2 + \frac{1}{2^n}} + x_n}$$

and

$$\sqrt{x_n^2 + \frac{1}{2^n}} + x_n \ge \sqrt{x_n^2} + x_n \ge \sqrt{1} + 1 = 2$$

we get

$$0 \leq x_{n+1} - x_n = \frac{\frac{1}{2^n}}{\sqrt{x_n^2 + \frac{1}{2^n}} + x_n} \leq \frac{1}{2^{n+1}} .$$

On the other hand, we have

$$x_{n+h} - x_n = (x_{n+h} - x_{n+h-1}) + (x_{n+h-1} - x_{n+h-2}) + \cdots + (x_{n+1} - x_n)$$

so

$$x_{n+h} - x_n \leq \frac{1}{2^{n+h}} + \frac{1}{2^{n+h-1}} + \cdots + \frac{1}{2^{n+1}} = \frac{1}{2^{n+1}} \left( \frac{1}{2^{h-1}} + \cdots + \frac{1}{2} + 1 \right)$$

which implies

$$x_{n+h} - x_n \leq \frac{1}{2^{n+1}} \left( \frac{1 - \frac{1}{2^h}}{1 - \frac{1}{2}} \right) \leq \frac{1}{2^n} .$$

Since $\{\frac{1}{2^n}\}$ converges to 0, then for any $\varepsilon > 0$, there exists $N_0 \geq 1$ such that for any $n \geq N_0$, we have $\frac{1}{2^n} < \varepsilon$ which implies $x_{n+h} - x_n < \varepsilon$ for any $n \geq N_0$ and any $h \geq 1$. This obviously implies that $\{x_n\}$ is Cauchy. Therefore, $\{x_n\}$ is convergent. Note that if we are able to prove that $\{x_n\}$ is bounded, then we will get again the same conclusion without the complicated algebraic calculations.

**Solution 3.21**

1. One can easily show that $I_0 = \pi/2$ and $I_1 = 1$. For $n \geq 2$, we use the integration by parts technique to show

$$I_{n+2} = \int_0^{\pi/2} \cos^{n+1}(t) \cos(t) dt = \Big[ \cos^{n+1}(t) \sin(t) \Big]_0^{\pi/2} + (n+1) \int_0^{\pi/2} \cos^n(t) \sin^2(t) dt,$$

   which implies $I_{n+2} = (n+1)\Big( I_n - I_{n+2} \Big)$ or

$$I_{n+2} = \frac{n+1}{n+2} I_n.$$

   Hence

$$I_{2n} = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{1}{2} I_0 = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2} = \frac{(2n)!\pi}{2^{2n+1}(n!)^2},$$

   and

$$I_{2n+1} = \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots \frac{2}{3} I_1 = \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots \frac{2}{3} = \frac{2^{2n}(n!)^2}{(2n+1)!}.$$

2. Note that since $0 \leq \cos^{n+1}(t) \leq \cos^n(t)$, for any $t \in [0, \pi/2]$, then $I_{n+1} \leq I_n$, i.e., $\{I_n\}$ is decreasing. In particular, we have $I_{n+2} \leq I_{n+1} \leq I_n$ and since $I_n > 0$ we get

$$1 \leq \frac{I_{n+1}}{I_{n+2}} \leq \frac{I_n}{I_{n+2}} = \frac{n+2}{n+1}.$$

   Hence $\lim\limits_{n \to \infty} \frac{I_{n+1}}{I_n} = 1.$

3. Since

$$(n+2)I_{n+1}I_{n+2} = (n+1)I_nI_{n+1}$$

we conclude that $\{(n+1)I_nI_{n+1}\}$ is a constant sequence. Hence

$$(n+1)I_nI_{n+1} = I_0I_1 = \frac{\pi}{2},$$

which implies $\lim\limits_{n\to\infty} 2nI_n^2 = \lim\limits_{n\to\infty} 2(n+1)I_nI_{n+1} = \pi$, or

$$\lim_{n\to\infty} I_n\sqrt{2n} = \sqrt{\pi}.$$

**Solution 3.22**

1. Note that $x_n > 0$ for $n \geq 1$. We have

$$\ln(x_{n+1}) - \ln(x_n) = \ln\left(\frac{x_{n+1}}{x_n}\right) = \ln\left(\frac{(n+1)!}{n!}\cdot\sqrt{\frac{n}{n+1}}\cdot e\cdot\frac{n^n}{(n+1)^{n+1}}\right)$$

which leads to

$$\ln(x_{n+1}) - \ln(x_n) = 1 - \left(n+\frac{1}{2}\right)\ln\left(1+\frac{1}{n}\right).$$

Note that we have

$$\lim_{n\to\infty} n^2\Big(\ln(x_{n+1}) - \ln(x_n)\Big) = \frac{1}{12}.$$

Indeed, using the Taylor approximation of $\ln(1+x)$ we get

$$\ln\left(1+\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{6n^3} + \frac{\varepsilon_n}{n^3}$$

where $\{\varepsilon_n\}$ goes to 0 when $n\to\infty$. Hence

$$\ln(x_{n+1}) - \ln(x_n) = 1 - \left(n+\frac{1}{2}\right)\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{6n^3} + \frac{\varepsilon_n}{n^3}\right) = -\frac{1}{6n^2} + \frac{1}{4n^2} - \frac{\varepsilon_n}{n^2} - \frac{\varepsilon_n}{2n^3}$$

which implies

$$\lim_{n\to\infty} n^2\Big(\ln(x_{n+1}) - \ln(x_n)\Big) = -\frac{1}{6} + \frac{1}{4} = \frac{1}{12}.$$

Since the series $\sum 1/n^2$ is convergent, the limit test will force $\sum \ln(x_{n+1}) - \ln(x_n)$ to be convergent. Hence $\ln(x_n)$ is convergent which in turn will force $\{x_n\}$ to be convergent. Set $l = \lim\limits_{n\to\infty} x_n = e^L$, where $L = \lim\limits_{n\to\infty} \ln(x_n)$. In particular, we have $l > 0$.

2. From the first part, we get

$$n! \approx l\left(\frac{n}{e}\right)^n\sqrt{n}, \text{ when } n\to\infty.$$

Using Wallis integrals (see Problem 3.21), $I_n = \displaystyle\int_0^{\pi/2}\cos^n(t)dt$, we know that $\lim\limits_{n\to\infty} I_n\sqrt{2n} = \sqrt{\pi}$, or

$$I_n \approx \sqrt{\frac{\pi}{2n}}, \text{ when } n\to\infty.$$

Since $I_{2n} = \dfrac{(2n)!\pi}{2^{2n+1}(n!)^2}$, we get

$$\sqrt{\frac{\pi}{4n}} \approx \frac{(2n)!\pi}{2^{2n+1}(n!)^2}, \quad \text{when } n \to \infty,$$

which implies

$$\sqrt{\frac{\pi}{4n}} \approx \frac{l(2n)^{2n}e^{-2n}\sqrt{2n}}{2^{2n}(\ln^n e^{-n}\sqrt{n})^2}\frac{\pi}{2}, \quad \text{when } n \to \infty.$$

Easy algebraic manipulations will lead to $l = \sqrt{2\pi}$.

3. Putting all the above results together we get

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}, \quad \text{when } n \to \infty.$$

**Solution 3.23**

- Notice that for any fixed $n$, $x_n = 2 + \frac{1}{2^n}$ if $n$ is even and $x_n = \frac{1}{2^n}$ if $n$ is odd. Thus $y_n = \sup\{x_n : \; k \geq n\} = 2 + \frac{1}{2^n}$ if $n$ is even and $2 + \frac{1}{2^{n+1}}$ if $n$ is odd. Hence

$$\limsup\{x_n\} = \inf\{y_n : \; n \in \mathbb{N}\} = 2.$$

  A similar calculation yields $\liminf\{x_n\} = 0$.

- Because $\{x_n\}$ is not bounded above, the limit superior does not exist. For the limit inferior, consider $z_n = \inf\{x_k : \; k \geq n\}$. Clearly, $z_n = x_n = 2^n$, since $\{x_n\}$ is monotone increasing and $z_n$ diverges to $\infty$. Thus supremum over $\{z_n : \; n \in \mathbb{N}\}$ does not exist, therefore the limit inferior does not exist. Note that even though the sequence $\{x_n\}$ is bounded below, limit inferior does not exist.

**Solution 3.24**

Since

$$\liminf_{n\to\infty} -x_n = -\limsup_{n\to\infty} x_n,$$

we will only prove the existence of a subsequence which converges to $\liminf_{n\to\infty} x_n$. It is clear that $\liminf_{n\to\infty} x_n = l \in \mathbb{R}$ since $\{x_n\}$ is bounded below. For any $\varepsilon > 0$ there exists $N \in \mathbb{N}$, such that for any $n \geq N$ we have

$$l - \varepsilon < \inf\{x_k; k \geq n\} \leq l.$$

Set $\varepsilon = 1$, then there exists $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$ we have

$$l - 1 < \inf\{x_k; k \geq n\} \leq l.$$

By induction one will construct an increasing sequence of integers $\{N_i\} \in \mathbb{N}$ such that for any $n \geq N_i$ we have

$$l - \frac{1}{i} < \inf\{x_k; k \geq n\} \leq l.$$

In particular, we have $l - 1/k < x_{N_k} \leq l$, which implies $\{x_{N_k}\} \to l$.

**Solution 3.25**

Note that for any sequence $\{x_n\}$ we have $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$. Since $\liminf_{n\to\infty} -x_n = -\limsup_{n\to\infty} x_n$, we will only show that $\liminf_{n\to\infty} x_n \leq \liminf_{n_k\to\infty} x_{n_k}$. By definition we have

$$\inf\{x_k; k \geq n\} \leq \inf\{x_{n_k}; n_k \geq n\}, \quad n \in \mathbb{N}.$$

Hence

$$\inf\{x_k; k \geq n'\} \leq \sup_{n\in\mathbb{N}} \Big(\inf\{x_{n_k}; n_k \geq n\}\Big), \quad n' \in \mathbb{N},$$

or

$$\sup_{n'\in\mathbb{N}} \Big(\inf\{x_k; k \geq n'\}\Big) \leq \sup_{n\in\mathbb{N}} \Big(\inf\{x_{n_k}; n_k \geq n\}\Big)$$

which implies $\liminf_{n\to\infty} x_n \leq \liminf_{n_k\to\infty} x_{n_k}$. Moreover if we assume that $\{x_{n_k}\}$ is convergent, then we have

$$\liminf_{n_k\to\infty} x_{n_k} = \limsup_{n_k\to\infty} x_{n_k} = \lim_{n_k\to\infty} x_{n_k},$$

which implies $\liminf_{n\to\infty} x_n \leq \lim_{n_k\to\infty} x_{n_k} \leq \limsup_{n\to\infty} x_n$. The converse is not true. Indeed, consider the sequence $\{(-1)^n\}$. Then we have $\liminf_{n\to\infty}(-1)^n = -1$ and $\limsup_{n\to\infty}(-1)^n = 1$. On other hand there does not exist a subsequence which converges to 0.

**Solution 3.26**

For any $N \in \mathbb{N}$, we have

$$x_n + y_n \leq \sup\{x_k;\ k \geq N\} + \sup\{y_k;\ k \geq N\},\ n \geq N$$

which implies $\sup\{x_n + y_n;\ n \geq N\} \leq \sup\{x_k;\ k \geq N\} + \sup\{y_k;\ k \geq N\}$. Hence

$$\inf_{N\in\mathbb{N}} \Big(\sup\{x_n + y_n;\ n \geq N\}\Big) \leq \inf_{N\in\mathbb{N}} \Big(\sup\{x_n;\ n \geq N\}\Big) + \inf_{N\in\mathbb{N}} \Big(\sup\{y_n;\ n \geq N\}\Big),$$

or $\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$. The equality does not hold in general. Indeed, we have $\limsup_{n\to\infty}(-1)^n = 1$, and $\limsup_{n\to\infty}(-1)^{n+1} = 1$, but $\limsup_{n\to\infty}(-1)^n + (-1)^{n+1} = 0$.

**Solution 3.27**

Assume first that $\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} = l \in \mathbb{R}$. So for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$, we have $l - \varepsilon \leq \inf_{n\geq N} \frac{x_{n+1}}{x_n}$, which implies $(l - \varepsilon)x_n \leq x_{n+1}$ for any $n \geq N$. This clearly implies $(l - \varepsilon)^{n-N} x_N \leq x_n$, for any $n \geq N$. Hence

$$(l - \varepsilon)^{(n-N)/n} x_N^{1/n} \leq x_n^{1/n}.$$

Since $(l - \varepsilon)^{(n-N)/n} x_N^{1/n} \to (l - \varepsilon)$ when $n \to \infty$, we get

$$l - \varepsilon \leq \liminf_{n\to\infty} x_n^{1/n}.$$

Since $\varepsilon$ was arbitrarily positive, we get

$$\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} \leq \liminf_{n\to\infty} \sqrt[n]{x_n}.$$

A similar proof will lead to

$$\limsup_{n\to\infty} \sqrt[n]{x_n} \leq \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

If $\{x_{n+1}/x_n\}$ is convergent, then we have

$$\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} = \limsup_{n\to\infty} \frac{x_{n+1}}{x_n},$$

which obviously implies

$$\liminf_{n\to\infty} \sqrt[n]{x_n} = \limsup_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \sqrt[n]{x_n}.$$

The converse is not true. Indeed, take $x_n = 2 + (-1)^n$, $n \in \mathbb{N}$. It is easy to check that $\sqrt[n]{x_n} \to 1$ when $n \to \infty$. But

$$\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{1}{3}, \text{ and } \limsup_{n\to\infty} \frac{x_{n+1}}{x_n} = 3.$$