1. Review about differentiation:

- Def: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$.

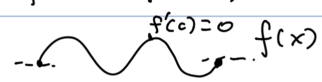
- Rmk: it may be that $f'(x)$ exists everywhere, but f' is not ~~cont~~ continuous.

$$f(x) = \begin{cases} x^2 \cdot \sin(\frac{1}{x}) & x > 0 \\ 0 & x \le 0 \end{cases}$$

$$f'(x) = \begin{cases} x^2 (-\frac{1}{x^2}) \cos(\frac{1}{x}) + 2x \cdot \sin\frac{1}{x}, & x > 0 \\ 0 \leftarrow \text{using definition.} & x = 0 \quad \because \lim_{h \to 0} \frac{f(h) - f(0)}{h - 0} = 0. \\ 0 & x < 0 \end{cases}$$

- Mean Value Thm: if $f: [a,b] \to \mathbb{R}$ cts, f' exist on (a,b).

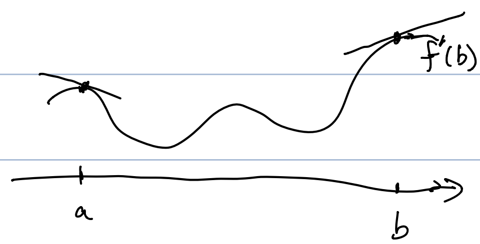
Then ① if $f(a) = f(b)$, then $\exists\, c \in (a,b)$, s.t. $f'(c) = 0$.



② Generalized MVT.

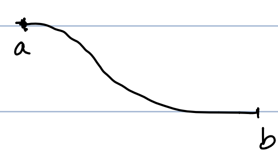
③ (common) MTV. $f(b) - f(a) = (b-a) \cdot f'(c)$.

- Intermediate Value Thm:



if $f'(a) < f'(b)$, then $\forall\, \mu$ strictly between $f'(a)$ and $f'(b)$.
$\exists\, c \in (a,b)$, s.t. $f'(c) = \mu$.

Q: if $f'(a) = f'(b) = 0$, is it possible to have a $c \in (a,b)$, such that $f'(c) = 0$?



$f'(c) < 0$, $\forall\, c \in (a,b)$

Ans: No.

④ L'Hôpital rule:

Today: Higher derivatives, Taylor expansion. Taylor Thm. Power Series.

Def (Higher derivative).

- If $f:[a,b]\to\mathbb{R}$ has derivatives $\forall x\in[a,b]$, and if $f'(x)$ is again differentiable at all pts, then we define

$$f''(x) = (f')'(x).$$

Similarly, the n-th derivative $f^{(n)}(x) := \left(f^{(n-1)}\right)'(x)$ if the RHS exist.

- short notations: $f^{(0)} = f$, $f^{(1)} = f'$, $f^{(2)} = f''$, $f^{(3)}$ ....

Ex: (1) $f(x) = x^n$, $f'(x) = n\cdot x^{n-1}$, $f''(x) = n(n-1)\cdot x^{n-2}$, ...

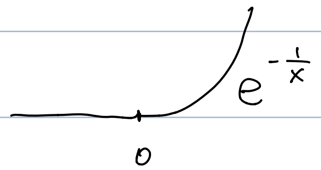
$n\in\mathbb{Z}$. (actually n can be any real numbers, in that case, we restrict $x>0$. e.g. $x^{\frac{1}{2}} = \sqrt{x}$)

(2). $f(x) = e^x$, $f'(x) = e^x$, ·· $f^{(n)}(x) = e^x$.

(3). $\sin x \underset{①}{\xrightarrow{(\cdot\cdot)'}} \cos x \underset{②}{\xrightarrow{(\cdot\cdot)'}} -\sin x \underset{③}{\xrightarrow{(\cdot\cdot)}} -\cos x \underset{④}{\xrightarrow{(\cdot\cdot)}} \sin x.$

Those are examples, where derivatives exists to all order.
i.e. $\forall x\in \text{domain}(f)$, $f^{(n)}(x)$ exists $\forall n\in\mathbb{N}$. We call such functions. smooth functions.

More Ex : (1) $f(x) = \begin{cases} 0 & x \le 0 \\ e^{-\frac{1}{x}} & x > 0 \end{cases}$



I claim : $f(x)$ is smooth $\forall\, x \in \mathbb{R}$. It is clear $\forall\, x \neq 0$.

We need to check that $f^{(n)}(0)$ exists, and equals $\lim_{x\to 0} f^{(n)}(x) = 0$. $\because \lim_{x\to 0^-} f^{(n)}(x) = 0$. $\therefore$ we need to show

that $f^{(n)}(0) = 0$, and $\lim_{x\to 0^+} f^{(n)}(x) = 0$

for $x > 0$,

$$f'(x) = \left(e^{-\frac{1}{x}}\right)' = \left(-\frac{1}{x}\right)' \cdot \left(e^{-\frac{1}{x}}\right) = \frac{1}{x^2} \cdot e^{-\frac{1}{x}}$$

$$\lim_{x\to 0^+} \frac{1}{x^2} \cdot e^{-\frac{1}{x}} = 0. \qquad \begin{array}{l} e^{-\frac{1}{x}} \to 0 \\ x^2 \to 0. \end{array}$$

Lemma : $\forall\, k \in \mathbb{N}$, $\lim_{u\to\infty} u^k \cdot e^{-u} = 0$.

pf: • refer to the section on seq. & limit

$$\cdot\ 0 \le \lim_{u\to\infty} \frac{u^k}{e^u} = \lim_{u\to\infty} \frac{u^k}{1 + u + \frac{u^2}{2!} + \cdots + \frac{u^k}{k!} + \frac{u^{k+1}}{(k+1)!} + \cdots}$$

$$\le \lim_{u\to\infty} \frac{u^k}{1 + \cdots + \frac{u^k}{k!} + \frac{u^{k+1}}{(k+1)!}} \le \lim_{u\to\infty} \frac{u^k}{\frac{u^{k+1}}{(k+1)!}} = 0$$

Apply this Lemma to $\lim_{x\to 0^+} \frac{1}{x^2} e^{-\frac{1}{x}} \underset{u=\frac{1}{x}}{=} \lim_{u\to\infty} u^2 \cdot e^{-u} = 0.$

Hence $\lim_{x\to 0^+} f'(x) = 0.$

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h - 0} = \begin{cases} \lim_{h\to 0^-} \frac{f(h) - f(0)}{h - 0} = 0 \\ \end{cases}$$

$\left(\lim_{h \to 0^+} \frac{f(h) - f(0)}{h - 0} = \lim_{h \to 0^+} \frac{1}{h} \cdot e^{-\frac{1}{h}} = 0. \right.$

Indeed.
$$f'(x) = \begin{cases} 0 & x \le 0 \\ \frac{1}{x^2} \cdot e^{-\frac{1}{x}}. & x > 0 \end{cases}$$

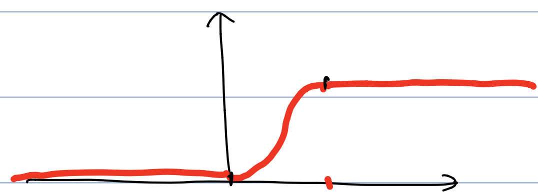
In general, one can prove the following by induction.

$$f^{(n)}(x) = \begin{cases} 0 & x \le 0. \\ \underbrace{P_n(\frac{1}{x}) \cdot e^{-\frac{1}{x}}}_{\text{polynomial. in } \frac{1}{x}} & x > 0. \end{cases}$$

#.

(2). "smooth step function"

it is a ~~fn~~ smooth fcn



$$g(x) = \begin{cases} 0 & x \le 0 \\ \in (0,1). & x \in (0,1) \\ 1 & x \geqslant 1 \end{cases}$$

Here is concrete construction:
$$g(x) = \frac{f(x)}{f(x) + f(1-x)}$$

To see that $\underline{g(x) \text{ is smooth}}$, we note both denominator & numerator are smooth, and denominator is nowhere vanishing.

Lemma: if $\alpha(x)$, $\beta(x)$ are smooth on $\mathbb{R}$, and. $\beta(x) \neq 0$ then $\alpha(x) / \beta(x)$ is smooth.

Pf: $\left(\frac{\alpha}{\beta}\right)' = \frac{\alpha'\beta - \beta'\alpha}{\beta^2}$ Let $\alpha_1 = \alpha'\beta - \beta'\alpha$, $\beta_1 = \beta^2$.

then α_1, β_1 are smooth, and $\beta_1(x) \neq 0$. $\forall x$.

$\rightsquigarrow$ define α_n, β_n recursively as $\left(\frac{\alpha_{n-1}}{\beta_{n-1}}\right)' = \frac{\alpha_n}{\beta_n}$.

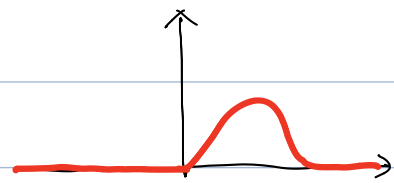
so. $(\alpha/\beta)^{(n)}$ exists for all $n \in \mathbb{N}$.

$$g(x) = \frac{f(x)}{f(x) + f(1-x)}$$

$\because f(x) = 0.$

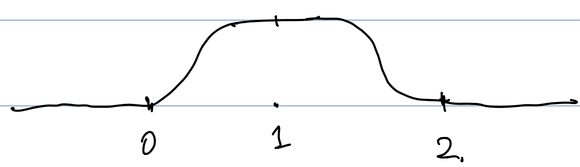
$$g(x) = \begin{cases} 0 & x \leq 0 \\ \in (0,1) & x \in (0,1) \\ = \frac{f(x)}{f(x)+0} = 1 & x \geq 1 \quad \because f(1-x) = 0 \end{cases}$$

(3). "smooth bump"

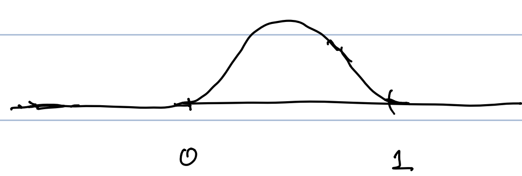


$$g(x) = \begin{cases} 0 & x \leq 0 \\ > 0 & x \in (0,1) \\ 0 & x \geq 1 \end{cases}$$

One way to construction: merge 2 smooth steps.



then rescale horizontally, to make it fit over $[0,1]$.



Another method:

$$g(x) = \frac{f(x)\, f(1-x)}{f(x) + f(1-x)}.$$

Taylor Theorem:

Non. rigorous statement:

Say $f(x)$ is smooth $\forall x \in \mathbb{R}$, and pick any $\alpha \in \mathbb{R}$.

then $$f(x) = \sum_{n=0}^{\infty} f^{(n)}(\alpha) \cdot \frac{1}{n!} \cdot (x-\alpha)^n. \qquad (*)$$

This works for $f(x) = e^x$, $\sin x$, x^n, ...

But is this true for all smooth functions?

Ans: No. take $f(x)$ to be the $\begin{cases} 0 & x \le 0 \\ e^{-\frac{1}{x}} & x > 0. \end{cases}$

take ~~then~~ $\alpha = 0$. Then $f^{(n)}(\alpha) = 0$, $\forall n = 0, 1, 2, \cdots$

Hence $$\sum_{n=0}^{\infty} f^{(n)}(\alpha) \frac{1}{n!} (x-\alpha)^n = 0. \qquad \forall x \in \mathbb{R}$$

but certainly $f(x) \ne 0$ for $x > 0$ so, $(*)$ is not true. in this case.

Thm: Let $f: [a,b] \to \mathbb{R}$ be a function, such that $f^{(n-1)}(x)$ exists and continuous on $[a,b]$, and $f^{(n)}(x)$ exists on (a,b). Then for any $\alpha, \beta \in [a,b]$, we have.

$$f(\beta) = f(\alpha) + f'(\alpha)\cdot(\beta-\alpha) + \frac{f''(\alpha)}{2!}(\beta-\alpha)^2 + \cdots + \frac{f^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^{n-1}$$

$$+ R_n(\alpha,\beta).$$

where $$R_n(\alpha,\beta) = \begin{cases} 0 & \text{if } \alpha = \beta \\ \dfrac{f^{(n)}(\gamma)}{n!}(\beta-\alpha)^n & \text{if } \alpha \ne \beta. \text{ for some, } \gamma \in (\alpha,\beta) \text{ or } \gamma \in (\beta,\alpha) \end{cases}$$

If we introduce notation

$$P_{\alpha,m}(x) = \sum_{k=0}^{m} \frac{f^{(k)}(\alpha)}{k!} (x-\alpha)^k$$

the " m-th " order Taylor expansion of f at α,

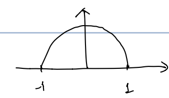
then the above is

$$f(x) - P_{\alpha,n-1}(x) = \frac{f^{n}(\gamma)}{n!} (x-\alpha)^n. \quad \text{for some } \gamma. \text{ between } x \text{ and } \alpha.$$

Given a smooth function $f(x)$ on $\mathbb{R}$

how to describe $f(x)$ near $x = \alpha$?

Ex: $f(x) = \sqrt{1-x^2}$. on $[-1, 1]$.



$f'(x)$ exists on $(-1, 1)$.

To the 0-th order, we use value of f at $x = \alpha$, i.e. $f(\alpha)$.

$$P_{\alpha,0}(x) = f(\alpha). \quad \text{const fcn.}$$

To the 1-st order, we use.

$$P_{\alpha,1}(x) = f(\alpha) + f'(\alpha)(x-\alpha) \quad \text{linear fcn.}$$

then $\begin{cases} P_{\alpha,1}(\alpha) = f(\alpha) \\ P'_{\alpha,1}(\alpha) = f'(\alpha) \end{cases}$

To the m-th order, if we know $f(\alpha), \cdots, f^{(m)}(\alpha)$.

then we can use $P_{\alpha,m}(x)$, a deg m polynomial in x,

s.t. $P^{(k)}_{\alpha,m}(\alpha) = f^{(k)}(\alpha) \quad k = 0, 1, \cdots, m.$

Q2: How to measure the error between the approximation & the actual function?

error of <u>const approx</u>

$$f(x) - P_{\alpha,0}(x) = f(x) - f(\alpha) \underset{\text{by MVT.}}{=} (x-\alpha)\cdot f'(\gamma)$$
for γ between x and α.

- error for linear approx ?

$$f(x) - P_{\alpha,1}(x) = f(x) - f(\alpha) - f'(\alpha)(x-\alpha) \overset{?}{=} (x-\alpha)^2 \cdot \frac{f''(\gamma)}{2!}$$
for $\gamma \in (x, \alpha)$.

Pf: · Prove the case that $x > \alpha$, $x \in [a, b]$. We will show that, $\exists\ \gamma \in (\alpha, x)$, s.t.
$$f(x) - P_{\alpha,n-1}(x) = \frac{f^{(n)}(\gamma)}{n!}(x-\alpha)^n.$$

· Let M be the const, such that
$$f(x) - P_{\alpha,n-1}(x) = \frac{M}{n!}(x-\alpha)^n.$$
(this M exists, since $x \neq \alpha$.)

· Define, $g(t) = f(t) - P_{\alpha,n-1}(t) - \frac{M}{n!}(t-\alpha)^n$.
(recall, α and x are fixed).

Then. $g(\alpha) = f(\alpha) - P_{\alpha,n-1}(\alpha) - 0 = 0$
$g(x) = f(x) - P_{\alpha,n-1}(x) - \frac{M}{n!}(x-\alpha)^n = 0$ $\because$ by choice of M

Also, for $k = 1, 2, \cdots, n-1$.
$$g^{(k)}(\alpha) = f^{(k)}(\alpha) - P^{(k)}_{\alpha,n-1}(\alpha) - \frac{M}{n!}\, n(n-1)\cdots(n-1+k)\cdot(\alpha-\alpha)^{n-k}$$
$$= 0.$$

Then $g(\alpha) = g(x) = 0 \overset{\text{MVT.}}{\Rightarrow} \exists\ \gamma_1 \in (\alpha, x)$, s.t. $g'(\gamma_1) = 0$

$g'(\alpha) = g'(\gamma_1) = 0 \Rightarrow \exists\ \gamma_2 \in (\alpha, x)$, s.t. $g^{(2)}(\gamma_2) = 0$

$\vdots$

$$g^{(n-1)}(\alpha) = g^{(n-1)}(\gamma_{n-1}) = 0 \Rightarrow \exists\, \gamma_n \in (\alpha, \gamma_{n-1}), \text{ s.t. } g^{(n)}(\gamma_n) = 0.$$

Now, $g^{(n)}(t) = f^{(n)}(t) - \cancel{P^{(n)}_{\alpha,n-1}}(t) - M$ $\quad = 0 \because \deg P \leq n-1.$

$\therefore \quad g^{(n)}(\gamma_n) = 0 = f^{(n)}(\gamma_n) - M.$

Thus, let $\gamma = \gamma_n$, we are done.