Last time:

- Taylor expansion. (finite term expansion with remainder)

  Taylor series (infinite terms sum. no remainder).

- For simplicity, assume that $f: [a,b] \to \mathbb{R}$ is smooth.

  (i.e $\forall x \in [a,b]$, $\forall n \in \{1,2,\cdots\}$, $f^{(n)}(x)$ exists).

"N-th order. Taylor expansion of f at (base) point $x_0 \in [a,b]$.

is. $P_{x_0,N}(x) = \sum_{n=0}^{N} f^{(n)}(x_0) \cdot \frac{1}{n!} (x-x_0)^n$

Taylor Theorem says:

$$f(x) - P_{x_0,N}(x) = \frac{f^{(N+1)}(\xi)}{(N+1)!} (x-x_0)^{N+1}$$

for some $\xi \in (x, x_0)$ a number between x and x_0.

(of course, if $x = x_0$, then $f(x_0) = P_{x_0,N}(x_0)$).

- Taylor series: Let $N \to \infty$. We "formally" write.

$$P_{x_0}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n.$$

Taylor series. at x_0.

This is an example of power series., whose general form

is

(*): $\sum_{n=0}^{\infty} c_n \cdot (x-x_0)^n$. $c_n \in \mathbb{R}$

Q: for what value of x, is the above series. convergent?

Prop: (Thm 3.39 Rudin). Consider power series $\sum_n c_n z^n$.

put $\alpha = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$, Let $R = \frac{1}{\alpha}$.

(if $\alpha = 0$, then $R = +\infty$; if $\alpha = +\infty$, then $R = 0$).

Then, the series is convergent if $|z| < R$;

and the series is divergent if $|z| > R$.

Rmk: if $|z| = R$, we don't know, it depends.

Pf: Use the root test for absolute convergence.

If $|z| < R$, then $|c_n \cdot z^n|^{\frac{1}{n}} = |c_n|^{\frac{1}{n}} \cdot |z|$.

$\therefore \limsup_{n\to\infty} |c_n z^n|^{\frac{1}{n}} = \alpha \cdot |z| < 1$. $\therefore \sum |c_n z^n|$ is convergent

$\therefore \sum c_n z^n$ is convergent (absolute convergence $\Rightarrow$ conv.)

If $|z| > R$, one can show $|c_n z^n|$ does not converge to 0.

#.

Thus, we can talk about convergence of Taylor series.

Ex: $f(x) = \frac{1}{1+x^2}$, we want to find its Taylor series based at $x_0 = 0$.

- instead of computing $f^{(n)}(0)$, and use the formula
$$\sum \frac{f^{(n)}(0)}{n!} x^n.$$

we directly manipulate

$\forall\ |x^2| < 1$,
$$\frac{1}{1+x^2} \overset{\checkmark}{=} \frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 \cdots \qquad \alpha = -x^2.$$
$$= 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots$$
$$= 1 - x^2 + x^4 - x^6 \cdots + (-x^2)^n + \cdots$$

This is the Taylor expansion at $x_0 = 0$.

• Radius of convergence? This series make sense if $|x|<1$.

This series is divergent if $|x|>1$, since x^{2n} would not converge to 0. $\Rightarrow$ Radius of Convergence $R=1$.

• we can also use root test. $|C_n| = \begin{cases} 0 & n \text{ odd} \\ 1 & n \text{ even.} \end{cases}$

$\limsup_{n\to\infty} |C_n|^{\frac{1}{n}} = 1 = \alpha \qquad R = \frac{1}{\alpha} = 1.$

Say f is smooth function.

Warning: ① Even if $P_{x_0}(x)$ series is convergent for $|x-x_0|<R$, it does not mean

$$f(x) = P_{x_0}(x) \quad \text{for} \quad |x-x_0|<R.$$

Ex: $f(x)$ smooth, s.t.

$$f(0)=0, \quad f'(0)=0, \quad f''(0)=2, \quad f^{(3)}(0)=0, \cdots \underset{n>2}{f^{(n)}(0)=0}$$

there exist a function satisfies the above data: namely, x^2.

But there exists more than one smooth fcn. satisfying these.

$\boxed{x^2+\varphi(x).}$ $\qquad \varphi(x) = \begin{cases} 0 & x \le 0 \\ e^{-\frac{1}{x}} & x>0. \end{cases}$

this also satisfies the above.

smooth and $\varphi^{(n)}(0) = 0$, $\forall n=0,1,2,$

$x \in (a,b)$

Aside: • If a smooth function $f(x)$ satisfies the condition that, $\forall x_0 \in (a,b)$, $\exists r_0>0$, s.t.

$$f(x) = P_{x_0}(x) \quad \forall \ |x-x_0|<r_0.$$

then we say $f(x)$ is a (real) analytic function.

Ex: $\sin(x)$, $\cos(x)$, e^x, polynomials are real analytic fun. and "Combinations" ($+, -, f(g(x)), \cdots$)

• (N-th order) Taylor expansion is a way to approximate a smooth function near a given point. But the approximation is not uniform over the entire domain of f.

Other ways of approximation methods exist,

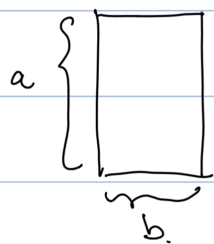
<u>Weierstrass Approximation Thm</u> : If f is a continuous function on $[a,b]$, then $\exists$ a sequence of polynomial $f_n(x)$, such $f_n \to f$ uniformly on $[a,b]$.

(See Ross for a proof).

Riemann - Stieltjes Integral (Rudin Ch 6).

• <u>Motivation</u>: measure area of some irregular shape.

rectangle



Area $= a \cdot b$.

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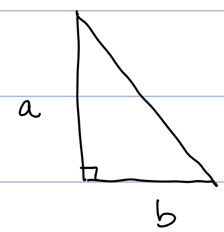
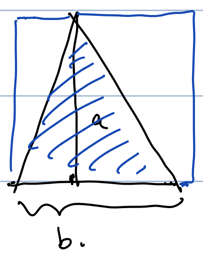


Area $= \frac{1}{2} ab$.



Area $= \frac{1}{2} a \cdot b$.

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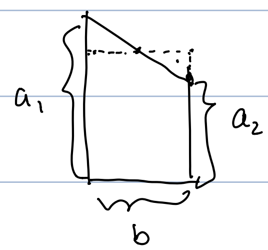
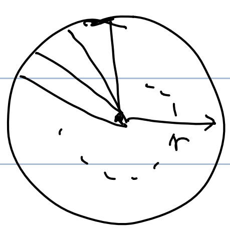
trapezoid.



Area $= \frac{a_1 + a_2}{2} \cdot b$.



Area $= \pi \cdot r^2$.

To derive this formula, one can cut the disk to "pizza slices.", and get

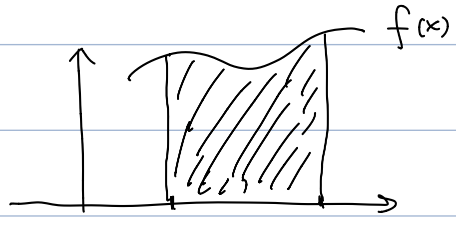
$$\text{area} = \frac{1}{2}(\text{circumference}) \cdot (\text{height of the slice})$$

$$= \frac{1}{2} \cdot (2\pi r) \cdot r = \pi r^2.$$

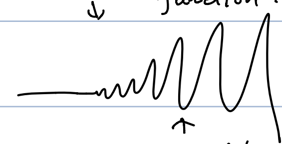
<u>General method</u>: cut the original shape in to smaller, but <u>"more standard"</u> shape, and add them up.

<u>Intuitively</u>: • if $f : [a,b] \to \mathbb{R}$ is a "<u>nice</u>" function,

then $\int_a^b f(x) \cdot dx$ is the area "under" the graph of f.



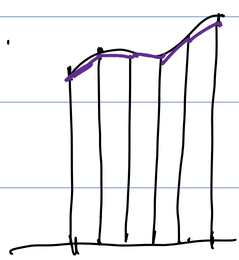
• is this a "nice" function?



• uniformly continuous?

• How to get approximations to <u>the "area"</u> under the graph?

One way to define area, is to get a sequence of approximations, and prove that the approximations (as the precision gets better & better), <u>converge</u> to something.



• cut the domain into equal size bins.

• replace the curve by trapezoid, and measure the area of each trapezoid, and take limit that bin size $\to 0$.

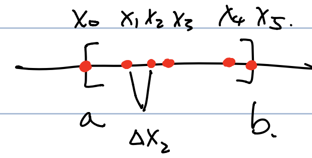
<u>Doubt</u>: (1) why do we need "equal size" cutting?

(2) why do we need the trapezoid approximation?

This is a plausible algorithm, but it is not a definition of the area, since it involves "arbitrary" choice.

- **Def**: (Partition) Let $[a,b] \subset \mathbb{R}$ be a closed interval. A partition P of $[a,b]$, is finite set of number in $[a,b]$:

$$a = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n = b$$



Define $\Delta x_i = x_i - x_{i-1}$, $i = 1, 2, \cdots, n$.

- We will consider $f : [a,b] \to \mathbb{R}$. real and bounded. (may not be continuous). standing assumption.

[if f is continuous, and f is defined on ~~a~~ $[a,b]$. then $f([a,b])$ is compact, hence bounded.

$$f(x) = \begin{cases} 0 & x \leq 0 \\ 1/x & x > 0. \end{cases} \quad \text{on } [-1,1].$$

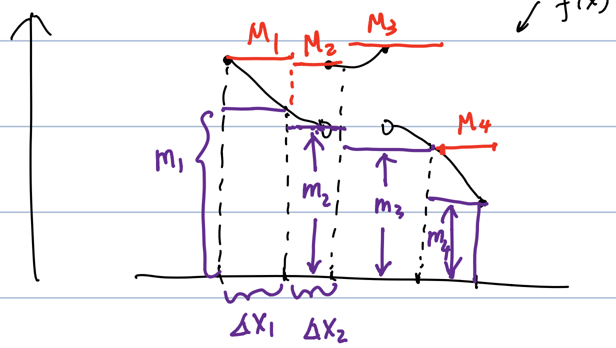
is a real valued function on $[-1,1]$, but unbounded.

Hence, we will NOT consider such function.

- Given $f : [a,b] \to \mathbb{R}$ bounded, and partition $P = \{x_0 \leq x_1 \leq \cdots \leq x_n\}$ we define

$$U(P,f) := \sum_{i=1}^{n} \Delta x_i \cdot M_i, \qquad M_i = \sup \{ f(x) \; ; \; x \in [x_{i-1}, x_i] \}$$

$$L(P,f) := \sum_{i=1}^{n} \Delta x_i \cdot m_i, \quad m_i = \inf\{f(x);\ x\in[x_{i-1},x_i]\}$$



$$\overline{\int_a^b} f\,dx$$

• <u>Definition</u>:

$$\mathcal{U}(f) := \inf_{P \text{ partition of } [a,b]} \mathcal{U}(P,f)$$

$$\underline{\int_a^b} f\,dx = L(f) := \sup_{P \text{ partition of } [a,b]} L(P,f).$$

Since f is bounded, hence, $\exists\ m, M \in \mathbb{R}$, s.t.

$m \le f(x) \le M \quad \forall x\in[a,b]$.

$\therefore\ \forall P$ partition of $[a,b]$,

$$\mathcal{U}(P,f) = \sum_{i=1}^{n} \Delta x_i \cdot M_i \le \sum_{i=1}^{n} \Delta x_i \cdot M$$

$$= M\cdot \sum_{i=1}^{n}\Delta x_i = M\cdot(b-a).$$

$$L(P,f) \ge m(b-a).$$

$$m(b-a) \le L(P,f) \le \mathcal{U}(P,f) \le M(b-a)$$

We say a function f is <u>Riemann integrable</u>, if

$\mathcal{U}(f) = L(f)$.

<u>Q</u>: Which functions are Riemann integrable over $[a,b]$?

Some sufficient conditions:

① if f is continuous, then $\int f\,dx$ exists.

$\Rightarrow$ $x \cdot \sin\left(\frac{1}{x}\right)$. over $[0,1]$ is integrable.

② if f is monotone, the $\int f\,dx$ exists.

⋮

<u>Next time</u>: · Riemann-Stieltje integral.

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