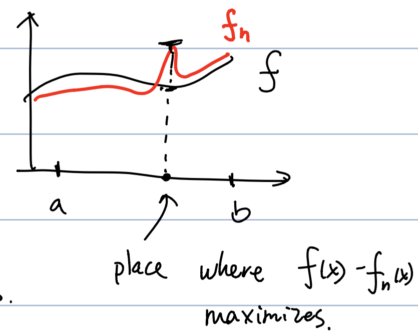
Today : 1. uniform convergence and $\frac{d}{dx}$, $\int$.

2. Review for the last 1/3 of the course.

(also, I'll be available for OH. during review week.).

Recall that : a sequence of functions $f_n : [a,b] \to \mathbb{R}$ is said to converge uniformly to a given function $f : [a,b] \to \mathbb{R}$ if $\lim_{n\to\infty} d_\infty(f, f_n) = 0.$, where $d_\infty(f, f_n) = \sup_{x\in[a,b]} |f(x) - f_n(x)|.$



Also recall : if $\{f_n\}$ are continuous and $f_n \to f$ uniformly, then f is continuous.

If $f_n \to f$ continuous.

Today : (1) and if f_n is integrable, is f integrable.? (Yes).

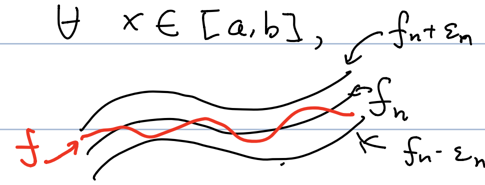
(2) and if f_n is differentiable, is f differentiable?
(No, ...)

Thm : (7.16 Rudin). Let α be monotone increasing on $[a,b]$.
Suppose $f_n \in \mathcal{R}(\alpha)$. , and $f_n \to f$ uniformly on $[a,b]$. Then f is integrable, and

$$\int_a^b f \, d\alpha = \lim_{n\to\infty} \int_a^b f_n \, d\alpha$$

Pf : Let $\varepsilon_n = \sup_x |f_n(x) - f(x)|$. Then $\forall x \in [a,b]$,

$$f_n(x) - \varepsilon_n \le f(x) \le f_n(x) + \varepsilon_n$$



Hence

$$\underline{\int} (f_n(x) - \varepsilon_n) dx \le \underline{\int} f \, d\alpha \le \overline{\int} f \, d\alpha \le \overline{\int} (f_n + \varepsilon_n) \, d\alpha$$

Note that

$$\overline{\int} f_n+\varepsilon_n \, d\alpha - \underline{\int} (f_n-\varepsilon_n)\, d\alpha$$
$$= \int (f_n+\varepsilon_n)\, d\alpha - \int (f_n-\varepsilon_n)\, d\alpha$$
$$= 2\varepsilon_n \cdot \int_a^b d\alpha = 2\varepsilon_n \cdot (\underbrace{\alpha(b)-\alpha(a)}).$$

goes to zero as $n\to\infty$ ↑ this is fixed

Hence

$$0 \le \overline{\int} f\, d\alpha - \underline{\int} f\, d\alpha \le 2\varepsilon_n(\alpha(b)-\alpha(a))$$

since $\varepsilon_n \to 0$ as $n\to\infty$, thus

$$\overline{\int} f\, d\alpha = \underline{\int} f\, d\alpha$$ and f is integrable w.r.t. α.

#.

Corrollary : Suppose $f_n \in \mathcal{R}(\alpha)$ on $[a,b]$, and.

$$F(x) = \sum_{n=1}^{\infty} f_n(x) \qquad \forall x\in[a,b]$$

the series converges uniformly (i.e. $F_N(x) = \sum_{n=1}^{N} f_n(x)$. and $F_N \to F$ uniformly), then $F \in \mathcal{R}(\alpha)$, and

$$\int_a^b F(x)\, d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n(x)\, d\alpha.$$

Pf : Since f_n are integrable, hence the finite sum of ~~it~~ them are integrable, hence F_N are integrable. Then we may apply the theorem.

$$\int_a^b F(x)\, d\alpha = \lim_{N\to\infty} \int_a^b F_N(x)\, dx$$
$$= \lim_{N\to\infty} \sum_{n=1}^{N} \int_a^b f_n(x)\, dx.$$

$\because \int (f+g)\, d\alpha = \int f\, d\alpha + \int g\, d\alpha$

$$=: \sum_{n=1}^{\infty} \int_a^b f_n(x)\, dx.$$

#,

Now, consider uniform convergence and differentiation. :

For example, consider

$$f_n(x) = \frac{1}{n} \cdot \sin(n^2 \cdot x).$$

Then $f_n \to 0$ uniformly on $\mathbb{R}$.

but $f_n'(x) = n \cdot \cos(n^2 x)$ fail to converge even pointwise

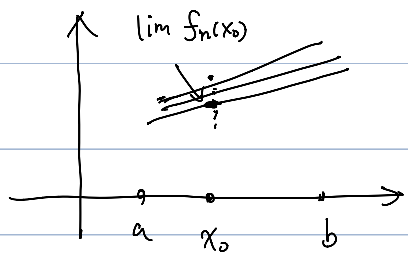
Theorem: Suppose $\{f_n\}$ is a sequence of differentiable functions on $[a,b]$, such that (a) $f_n'(x)$ converges uniformly to $g(x)$, and (b). $\exists\, x_0 \in [a,b]$, such that $\{f_n(x_0)\}$ converges.

Then, $f_n(x)$ converges to some function f uniformly, and.

$$f'(x) = g(x) \left(= \lim_{n\to\infty} f_n'(x) \right).$$

(Note here, we are not assuming $f_n'(x)$ is continuous.).

idea:



example:

$$f_n(x) = \left(3+\frac{1}{n}\right) + x \cdot \left(2+\frac{1}{n}\right).$$

$\because\ f_n'(x) = 2+\frac{1}{n}. \to 2.$

$f_n(0) = 3+\frac{1}{n}. \to 3$

$\therefore$ f_n converges uniformly on $[0,1]$

(not convergent unif on $\mathbb{R}$!)

Pf: Given $\varepsilon > 0$. Choose N large enough, such that the following conditions are satisfied:

(1) $$|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2}. \qquad \forall\, n, m > N.$$

(2) $\sup_{t\in[a,b]} |f_n'(t) - f_m'(t)| < \frac{\varepsilon}{2(b-a)}$ $(\forall\ n,m>N)$.

$\forall n,m>N$, to $f_n - f_m$ $f_{n,m}(x) = f_n(x) - f_m(x)$

Apply mean value theorem, $\forall\ x,t \in [a,b]$

(*). $\underbrace{\left| f_n(x) - f_m(x) - (f_n(t) - f_m(t)) \right|}_{= [f_n'(\xi) - f_m'(\xi)]\cdot(x-t)} \leq \frac{|x-t|\cdot\varepsilon}{2(b-a)} \leq \frac{\varepsilon}{2}$.

for some $\xi \in (x,t)$ or (t,x)

For any point $x \in [a,b]$, we have

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_m(x) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)|$$
$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus. $\{f_n(x)\}$ is uniformly Cauchy, hence converges uniformly on $[a,b]$. We may denote the limit as $f(x)$.

Now, let's fix a point $x \in [a,b]$, we are going to prove $f'(x) = \lim_{n\to\infty} f_n'(x)$.

We consider difference quotients:

$$\phi(t) = \frac{f(t)-f(x)}{t-x}, \qquad \phi_n(t) = \frac{f_n(t) - f_n(x)}{t-x}.$$

pointwise, $\forall\ t \neq x$, $\phi_n(t) \to \phi(t)$. but the $\frac{1}{t-x}$ factor may ruin the unif. conv.

Then for each n, since f_n are differentiable.

$$\lim_{t\to x} \phi_n(t) = f_n'(x).$$

Then (*) implies. $|\phi_n(t) - \phi_m(t)| \leq \frac{\varepsilon}{2(b-a)}$.

Hence, $\phi_n(t)$ converges uniformly for $t \neq x$. Since, $f_n \to f$,

we have $\phi_n \to \phi$ uniformly on $t\in[a,b]$, $t\neq x$.

Finally, by uniform convergence,

$$\lim_{n\to\infty} \lim_{t\to x} \phi_n(t) = \lim_{t\to x} \lim_{n\to\infty} \phi_n(t).$$

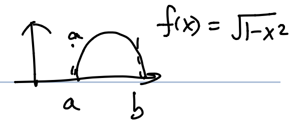
$$\Rightarrow \quad \lim_{n\to\infty} f_n'(x) = \lim_{t\to x} \phi(t) = f'(x). \qquad \#.$$

Review:

1. Differentiation. $\quad f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}.$

- "the slope" of f at point x.
- if $f'(x)$ exists at $x = x_0$, $\Rightarrow$ $f(x)$ is continuous at x_0.
- if $f'(x)$ exist over (a, b), and if $f(x)$ is continuous at $[a, b]$, then we have

example



mean value theorem:

$$f(b) - f(a) = (b-a) f'(\xi), \quad \text{for some } \xi \in (a, b).$$

it relate the difference quotient $\dfrac{f(b) - f(a)}{b-a}$, with the derivative $f'(\xi)$.

- Taylor theorem: if $f^{(n)}(x)$ exists on (a, b), and $f^{(n-1)}(x)$ is continuous on $[a, b]$, Then. $\forall x_0 \in [a, b]$, we can do an n-term Taylor expansion around x_0 with error term.

if $x \neq x_0$, $x \in [a, b]$

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0) + \cdots + \frac{f^{(n-1)}(x_0)}{(n-1)!}(x - x_0)^{n-1} + \frac{f^{(n)}(\xi)}{n!}(x - x_0)^n.$$

ξ is between x_0 and x.

(mean value theorem is a special case for $n=1$).

warning 1: One can have a smooth function $f(x)$, whose Taylor series.

$$P_{x_0}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n.$$

converges for x in a nbhd of x_0, say $|x-x_0|<\delta$.

But $f(x) \neq P_{x_0}(x)$ for $|x-x_0|<\delta$.

warning 2: For smooth functions, $f(x)$. the radius of convergence for Taylor series $P_{x_0}(x)$, may be zero.

i.e, if $x \neq x_0$, then the sum

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$$

is divergence.

(Borel Lemma: For any sequence $c_0, c_1, c_2, \cdots$ of real number. $\exists f(x)$ smooth, s.t. $f^{(n)}(0) = c_n \quad \forall n=0,1,2\ldots$)

Now, for integration:

- Riemann integral and Riemann – Stieltjes integral.

  (not all statement about Rieman integral is true for Rieman - Stieltjes integral).

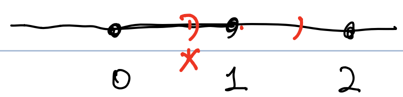
- Meaning of the "weight function" α. $I = [c, d]$.

  $\alpha(d) - \alpha(c)$ = α-weight of the interval I.

- Using partition and upper sum $U(P, f, \alpha)$ & lower sum $L(P, f, \alpha)$ to approximate upper integral $U(f, \alpha)$. & lower integral $L(f, \alpha)$

(by boundedness of f, $U(f,\alpha)$ and $L(f,\alpha)$ always exist.)

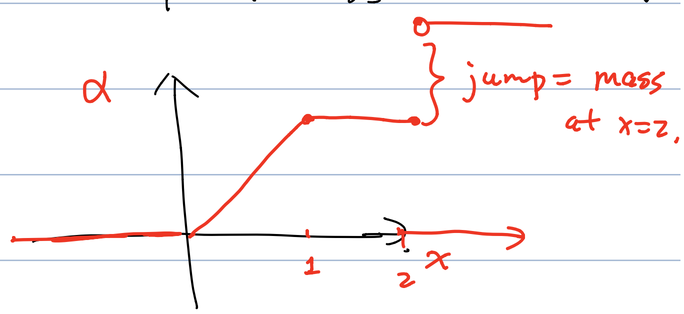
- Ex:



consider weight distribution on $\mathbb{R}$, two parts.

- uniform distribution on interval $[0,1]$
- a point mass at $x=2$ of mass $=1$

$\Rightarrow$ α <u>can be</u> taken as,



Results about integrable fcn:

- f is real and bounded function on $[a,b]$.

① If f is continuous, then f is $\mathcal{R}(\alpha)$ $\forall \alpha$.

② If f is monotone, and α is continuous (and increasing), then $f \in \mathcal{R}(\alpha)$.

③ If f has finitely many discontinuity,

if α is continuous at those discontinuities,
then $f \in R(\alpha)$.

(④ (not required, but nice to know): if α is continuous, and Discontinuity (f) has "measure zero", for example, countable, then $f \in R(\alpha)$.)

Lebesgue criterion. for Riemann integrability.