Recall: Mean Value Theorem for $f'(x)$.

Let $f, g : [a,b] \to \mathbb{R}$ continuous, and differentiable over (a,b).

Thm: There exists a point $c \in (a,b)$, such.

(Generalized MVT) $[f(b) - f(a)] \cdot g'(c) = [g(b) - g(a)] f'(c)$.

(MVT) $f(b) - f(a) = (b-a) \cdot f'(c)$. $g(x) = x$, $g'(x) = 1$

Proving this uses lemma that: if c is a local max (or local min) of a function, and if derivative exists at c, and c is an interior point $\Rightarrow$ derivative vanishes.

· Cor: if $f'(x) > 0$ $\forall x \in (a,b)$, then. f is a strictly increasing function on $[a,b]$.

Pf: We need to show that, $\forall \alpha < \beta$ in $[a,b]$, $f(\alpha) < f(\beta)$. Consider $f(\beta) - f(\alpha)$, if we apply MVT to $[\alpha, \beta]$, then.

$$[f(\beta) - f(\alpha)] = (\beta - \alpha) \cdot f'(\gamma) > 0$$

for some $\gamma \in (\alpha, \beta)$. #.

similarly.

Cor: if $f'(x) \geq 0$, ---, then $f(x)$ is a increasing func. over $[a,b]$

Cor: Suppose $f: \mathbb{R} \to \mathbb{R}$. is continuous, and $f'(x)$ exists $\forall x \in \mathbb{R}$.

~~and~~ Assume $\exists M > 0$, s.t. $|f'(x)| \leq M$. $\forall x$. Then.

f is unif cont.

Pf: $\forall \varepsilon > 0$, we may take $\delta = \frac{\varepsilon}{M}$. Then $\forall x < y$, with $|y - x| < \delta$, we have,

$$f(y) - f(x) = (y-x)\cdot f'(c) \quad \text{for some } c \in (x,y).$$

$$\Rightarrow \quad |f(y) - f(x)| = |y-x| \cdot |f'(c)| < \delta \cdot M = \varepsilon.$$

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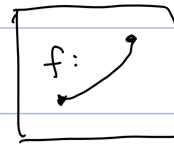
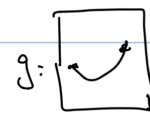
- Intermediate Value Theorem for $f'(x)$.

Thm: Let $f: [a,b] \to \mathbb{R}$ be a differentiable function.
Assume, $f'(a) < f'(b)$. Then for any $\mu \in \mathbb{R}$, with $f'(a) < \mu < f'(b)$, there exists a $c \in (a,b)$ such that $f'(c) = \mu$.



Pf: Let $g(x) = f(x) - \mu \cdot x$. Then



$g'(a) = f'(a) - \mu < 0, \quad g'(b) = f'(b) - \mu > 0.$

Let $c \in [a,b]$, s.t. $c = \min g([a,b])$.

Since $g'(a) < 0$ and $g'(b) > 0$, then a, b are not global minimum. $\Rightarrow c \in (a,b)$. Apply the Lemma (last time)

$g'(c) = 0 \iff f'(c) = \mu$.

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L'Hopital Rule.

Ex: $$\lim_{x\to 0} \frac{\sin(x)}{x} \overset{\text{L'Hoptal}}{=} \lim_{x\to 0} \frac{(\sin x)'}{(x)'} = \lim_{x\to 0} \frac{\cos x}{1} = \cos(0) = 1$$

$$\lim_{x\to\infty} \frac{\log x}{x} \overset{\text{L'H.}}{=} \lim_{x\to\infty} \frac{(\log x)'}{(x)'} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0.$$

in general, if $f(x) = (x-a)^n \cdot h(x)$, $\quad h(a) \neq 0$

$g(x) = (x-a)^n \cdot k(x)$, $\quad k(a) \neq 0$.

then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{h(a)}{k(a)}$.

(Non-example):

cannot apply, since $\lim_{x\to 0} \log x \neq 0$.

$$\lim_{x \to 0} \frac{\log x}{x} \neq \lim_{x\to 0} \frac{(\log x)'}{(x)'} = \lim_{x \to 0} \frac{1}{x} = +\infty.$$

$$\| \\ -\infty$$

Thm: Let $f, g : (a, b) \to \mathbb{R}$, differentiable.

Assume that $g(x) \neq 0$ over (a, b) (WLOG, $g(x) > 0$).

$g'(x) \neq 0$.

Assume that

(1) $\lim_{x \to a} \frac{f'(x)}{g'(x)} = C \in \mathbb{R} \cup \{+\infty, -\infty\}$.

and one of the following holds

(i) $\lim_{x \to a} f(x) = 0$, $\lim_{x\to a} g(x) = 0$.

(ii) $\lim_{x \to a} g(x) = +\infty$

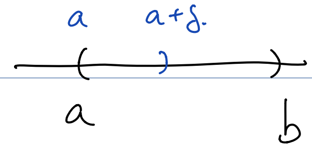
Then, we have $\lim_{x \to a} \frac{f(x)}{g(x)} = C$.

Pf: Consider the case $C \in \mathbb{R}$ ($C = \pm\infty$ as exercise).

$\because \lim_{x \to a} \frac{f'(a)}{g'(a)} = C$, then $\forall \varepsilon > 0$, $\exists \delta > 0$, such

that $\forall x \in (a, a+\delta)$, we have

$$C-\varepsilon < \frac{f'(x)}{g'(x)} < C+\varepsilon$$



- $\forall\ a < \alpha < \beta < a+\delta$, we can apply the generalized MVT, $\exists\ \gamma \in (\alpha, \beta)$, s.t.

$$[f(\beta) - f(\alpha)]\ g'(\gamma) = [g(\beta) - g(\alpha)] \cdot f'(\gamma).$$

$$\Rightarrow \quad \frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} = \frac{f'(\gamma)}{g'(\gamma)}. \quad \text{whenever } g(\beta)-g(\alpha) \neq 0.$$

$$\Rightarrow (*) \quad C-\varepsilon < \frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} < C+\varepsilon \quad \forall\ \alpha, \beta \in (a, a+\delta) \text{ and } g(\beta) \neq g(\alpha).$$

Case (i): $\lim_{x \to a} g(x) = 0$, $\lim_{x \to 0} f(x) = 0$

Take $(*)$, and let $\alpha \to a$. Since $g(\beta)$ is fixed (and $g(\beta) \neq 0$), and $g(\alpha) \to 0$, hence $g(\beta) \neq g(\alpha)$ for α close enough to a.

$$\Rightarrow \quad \lim_{\alpha \to a} \frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} = \frac{f(\beta)}{g(\beta)} \in [C-\varepsilon, C+\varepsilon]$$

Now, we have proved for any $\varepsilon > 0$, $\exists\ \delta > 0$, s.t. $\forall \beta \in (a, a+\delta)$, $\frac{f(\beta)}{g(\beta)} \in [C-\varepsilon, C+\varepsilon]$.

thus $\lim_{\beta \to a} \frac{f(\beta)}{g(\beta)} = C$.

Case (ii): $\lim_{x \to a} g(x) = +\infty$.

$$\frac{f(\alpha) - f(\beta)}{g(\alpha) - g(\beta)} = \frac{\frac{f(\alpha)}{g(\alpha)} - \frac{f(\beta)}{g(\alpha)}}{1 - \frac{g(\beta)}{g(\alpha)}}.$$

Send $\alpha \to a$, we get.

$$C - \varepsilon \leq \liminf_{\alpha \to a} \frac{f(\alpha)}{g(\alpha)} \leq \limsup_{\alpha \to a} \frac{f(\alpha)}{g(\alpha)} \leq C + \varepsilon$$

$$\liminf_{\alpha \to a} \frac{f(\alpha)}{g(\alpha)} = \liminf_{\alpha \to a} \frac{\frac{f(\alpha)}{g(\alpha)} - \frac{f(\beta)}{g(\alpha)}}{1 - \frac{g(\beta)}{g(\alpha)}} = \liminf_{\alpha \to a} \frac{f(\alpha) - f(\beta)}{g(\alpha) - g(\beta)}.$$

$$\limsup_{\alpha \to a} \frac{f(\alpha)}{g(\alpha)} = \limsup_{\alpha \to a.} \frac{f(\alpha) - f(\beta)}{g(\alpha) - g(\beta)}$$

Since ε is an arbitrary positive number, hence.

$$\liminf_{\alpha \to a} \frac{f(\alpha)}{g(\alpha)} = \limsup_{\alpha \to a} \frac{f(\alpha)}{g(\alpha)} = C = \lim_{\alpha \to a} \frac{f(\alpha)}{g(\alpha)}$$

#,

- Ex: • $\lim_{x \to \infty} \left(\frac{x^2}{e^{3x}} \right) = \lim_{x \to \infty} \frac{(x^2)'}{(e^{3x})'} = \lim_{x \to \infty} \frac{2x}{3e^{3x}} = \lim_{x \to \infty} \frac{(2x)'}{(3e^{3x})'}$

$$= \lim_{x \to \infty} \frac{2}{9 \cdot e^{3x}} = 0$$

by continuity of exp(---)

by $\lim x \log x = 0$

• $\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{\boxed{x \cdot \log x}} \overset{?}{=} e^0 = 1 \qquad x = e^{\log x}.$

$$\lim_{x\to 0} x\cdot\log x = \lim_{x\to 0}\frac{x}{\frac{1}{\log x}} = -\lim_{x\to 0}\frac{x}{\frac{-1}{\log x}} = -\lim_{x\to 0}\frac{(x)'}{\left(\frac{-1}{\log x}\right)'}$$

$$= -\lim_{x\to 0}\frac{1}{\left(\frac{1}{\log x}\right)^2\cdot\frac{1}{x}} = -\lim_{x\to 0} x\cdot(\log x)^2$$

$$\left(\frac{-1}{\log x}\right)' = \frac{1}{(\log x)^2}\cdot\frac{1}{x}$$

$$= \lim_{x\to 0}\frac{\log x}{\frac{1}{x}} = \lim_{x\to 0}\frac{(\log x)'}{\left(\frac{1}{x}\right)'} = \lim_{x\to 0}\frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0}(-x)$$

$$= 0.$$

- $$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty}\left[e^{\log\left(1+\frac{1}{x}\right)}\right]^x = \lim_{x\to\infty} e^{x\cdot\log\left(1+\frac{1}{x}\right)}$$

$$= e^1 = e.$$

$$\lim_{x\to\infty} x\cdot\log\left(1+\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\log\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

$$= \lim_{x\to\infty}\frac{\cancel{\left(-\frac{1}{x^2}\right)}\cdot\left(\frac{1}{1+\frac{1}{x}}\right)}{\cancel{\left(-\frac{1}{x^2}\right)}} = \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = 1,\quad \left(\log\left(1+\frac{1}{x}\right)\right)'$$

$$= \left(-\frac{1}{x^2}\right)\cdot\left(\frac{1}{1+\frac{1}{x}}\right)$$