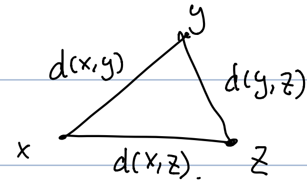
Today: §13. metric space and topology.

• **Def**: A metric space is a set S, together with a distance function $d: S \times S \to \mathbb{R}$, such that

① $d(x,y) \geq 0$, and $d(x,y) = 0 \iff x = y$.

② $d(x,y) = d(y,x)$.

③ $d(x,y) + d(y,z) \geq d(x,z)$



• **Ex**: (1). $S = \mathbb{R}$. $d(x,y) = |x-y|$

(2). $S = \mathbb{R}^2$. $\forall\ \vec{x} = (x_1, x_2) \in \mathbb{R}^2$, $\vec{y} = (y_1, y_2) \in \mathbb{R}^2$.

Euclidean distance:

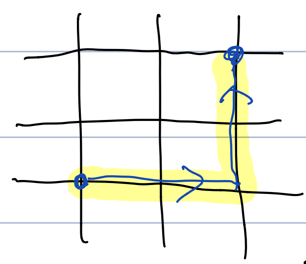
$$d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$$

(3) $S = \mathbb{R}^n$, $d(x,y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$

(4) "L^p-distance" $S = \mathbb{R}^n$. $1 \leq p < \infty$

$$d_p(x,y) = \left[\sum_{i=1}^n |x_i - y_i|^p\right]^{\frac{1}{p}}. \quad (*)$$

(5) taxi-cab-distance.



$S = \mathbb{R}^2$ (or in general $\mathbb{R}^n$)

$d(x,y) = \cancel{|x_2 - x_1| + |y_2 - y_1|}$ (L^1-distance)

$|x_1 - y_1| + |x_2 - y_2|$.

(6). "max"-distance. (L^∞-distance).

$S = \mathbb{R}^2$, $d_{max}(x,y) = \max\{\cancel{|x_2 - x_1|}, \cancel{|y_2 - y_1|}\}$.

$|x_1 - y_1|, |x_2 - y_2|$.

$$\lim_{p\to\infty} d_p(x,y) = d_{max}(x,y).$$

$\because$ if $\overset{|x_1-y_1|}{\overset{\parallel}{a}} > 0$, $\overset{|x_2-y_2|}{\overset{\parallel}{b}} > 0$, $\underline{a \geq b}$

$$(a^p + b^p)^{\frac{1}{p}} = \left[a^p \left(1 + \left(\frac{b}{a}\right)^p\right)\right]^{\frac{1}{p}}.$$

$$= a \cdot \Big(\underbrace{1 + \left(\tfrac{b}{a}\right)^p}_{\text{bounded. } 1 \leq \cdot \leq 2}\Big)^{\frac{1}{p}} \to a \cdot 1 \qquad 0 < \left(\frac{b}{a}\right) < 1$$

and $\forall M > 0$

$$\lim_{p \to \infty} M^{\frac{1}{p}} = 1.$$

To verify triangle inequality for d_{max}.

Take $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2)$.

WTS: $d(x,y) + d(y,z) \geq d(x,z)$.

$$d(x,z) = \max(|x_1 - z_1|, |x_2 - z_2|).$$

$\because$ $|x_1 - z_1| \leq |x_1 - y_1| + |y_1 - z_1|$

$|x_2 - z_2| \leq |x_2 - y_2| + |y_2 - z_2|$.

$\therefore$ $\max(|x_1 - z_1|, |x_2 - z_2|) \leq \max(|x_1 - y_1| + |y_1 - z_1|, |x_2 - y_2| + |y_2 - z_2|)$.

$$d(x,y) + d(y,z) = \max(|x_1 - y_1|, |x_2 - y_2|) + \max(|y_1 - z_1|, |y_2 - z_2|).$$

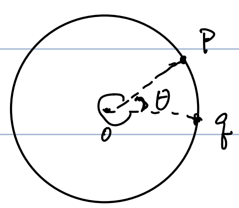
$\geq |x_1 - y_1| + |y_1 - z_1|$. ①

and $\geq |x_2 - y_2| + |y_2 - z_2|$. ②

$\Rightarrow$ $d(x,y) + d(y,z) \geq \max(|x_1 - y_1| + |y_1 - z_1|, |x_2 - y_2| + |y_2 - z_2|)$

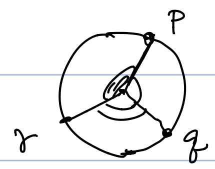
$\geq d(x,z)$. #

Ex: Let S be the unit circle in $\mathbb{R}^2$

$S =$ 

, how to define a distance function.

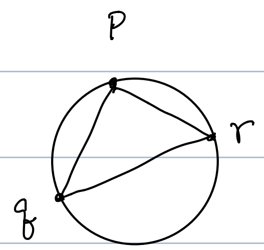
between 2 points? (1) $\theta(p,q)$. the angle that $\leq \pi$.



$d(p,q) + d(q,r) \geq d(p,r)$

equivalently, $d(p,q)$ = (arc length of the shorter arc connecting p, q.)

(2).



$d(p,q)$ = length of the interval (chord). connecting (p,q).

Ex: Can you define distance on $S^2 \subseteq \mathbb{R}^3$?

Cauchy Sequence in a metric space (S,d):

Def: A sequence $(S_n)_n$ in S is Cauchy if $\forall \varepsilon > 0$, there exists a $N > 0$. s.t. $\forall n, m > N$,

$$d(S_n, S_m) < \varepsilon.$$

Convergence: We say a seq $(S_n)_n$ in (S,d) converge to a point $s \in S$, if $\forall \varepsilon > 0$, $\exists N > 0$, s.t. $d(S_n, s) < \varepsilon$ $\forall n > N$.

Completeness: A metric space (S,d) is complete, if every Cauchy seq is convergent.

- Induced distance function: if (S,d) is a metric space, and $A \subset S$ is any subset, then (A,d) is a metric space.

Ex: (1) $S = \mathbb{Q}$. $d(x,y) = |x-y|$.

then (S,d) is not complete.

$\Leftrightarrow$ exist a Cauchy seq in S, but does not converge to an point $s \in \mathbb{Q}$.

Let (S_n) be a seq of rational number, convergent to an irrational number $\alpha \in \mathbb{R}$. then (S_n) is a Cauchy seq in S, but (S_n) doesn't converge to any rational number.

(2). $S = \mathbb{R} \setminus \{0\}$, $d(x,y) = |x-y|$.

(S,d) is not complete, since the seq $(\frac{1}{n})_n$ is Cauchy, but not convergent in S. $(\because 0 \notin S)$.

(3). $S = \mathbb{R}^n$ is a complete metric space.

Sketch of proof:

(1). Let $S_1, S_2, \cdots$ be a seq in $\mathbb{R}^n$, where $S_k \in \mathbb{R}^n$ has coordinate

$$S_k = (S_k^{(1)}, \cdots, S_k^{(n)}) \in \mathbb{R}^n.$$

then we can form n sequenes in $\mathbb{R}$: $1 \le i \le n$.

$$(S_m^{(i)})_m = S_1^{(i)}, S_2^{(i)}, \cdots \quad m \in \mathbb{N}.$$

Then. $(S_m)_m$ is a Cauchy seq in $\mathbb{R}^n$ iff $(S_m^{(i)})_m$ for each $1 \le i \le n$ is a Cauchy seq in $\mathbb{R}$.

(2). $(S_m)_m$ converges to $S \in \mathbb{R}^n$.

$\Leftrightarrow$ $\forall\ 1 \le i \le n$, $(S_m^{(i)})_m$ converges to $S^{(i)}$ in $\mathbb{R}$. #

recall the BW thm : every bound seq has convergent subseq.

Thm : (Bolzano - Weierstrass for $\mathbb{R}^n$).

Every bounded sequence $(S_m)_m \in \mathbb{R}^n$ has a convergent sub seq.

Pf : To construct a subseq convergent in $\mathbb{R}^n$

$\Leftrightarrow$ to get a subseq, whose i-th coord. converges, $\forall\ 1 \le i \le n$.

We can construct this sub.seq, step by step.

step 1 : take subseq of (S_m), s.t. $(S^{(1)}_{m_k})_k$ converges. Then, replace (S_m) by this subseq.

step 2 : take subseq of the new (S_m), s.t. $(S^{(2)}_{m_k})_k$ converges. Then, replace (S_m) by this subseq.

⋮

Step n : - - - ,
- - -

Then we get a subseq, such that $(S^{(i)}_m)_m$ converges $\forall 1 \le i \le n$. #

Topology :

Def : Let S be a set. A topological structure on S is the data of a collections τ of subsets in S,

(If $U \subset S$, and U is in τ, we say U is open.)

τ needs to satisfy

① S and ϕ are open subset

② arbitrary unions of open subsets is still open

③ finite intersections of open sets are open.

Ex: Topology on $\mathbb{R}$ $(-\infty, +\infty)$

- (a, b), (a, ∞), $(-\infty, a)$, are open subsets.
- $\bigcup_{n \in \mathbb{Z}} (n, n+\frac{1}{2})$. $\cdots (\;) \; (\;) \; (\;) \; (\;) \cdots$ is open.
- $\bigcap_{n \in \mathbb{N}} (-\frac{1}{n}, \frac{1}{n}) = \{0\}$ is not open.

Topology on metric space:

- (S, d) metric space. $A \subset S$. subset.

  $p \in A$ is called an interior point, if $\exists\, \varepsilon > 0$,

such $B_\varepsilon(p) := \{ q \mid d(q, p) < \varepsilon, q \in S \}$, is contained in A.

- $A \subset S$ is open, if every point in A is an interior point.