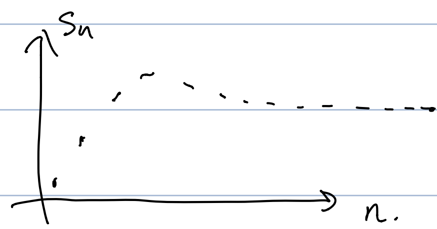
- Midterm: • Next Thursday in class (Feb 18th).
    - Content: upto this week's content. (not including Next Tuesday).
    - practice problems will be given.

- **Last time:** • subsequence (§11).
    - If S_n is a convergent sequence, then it has a monotone subseq.

Q: If $A = \limsup S_n \in \mathbb{R}$, is it always true that one can find an increasing sub-seq ~~to~~ convergents to A?

No. Example:



Today: We begin by this more general lemma:

Lemma: For any sequence (S_n), there exists a monotone subsequence.

Pf: We say a term S_n in the seq is dominant, if $\forall\, k > n$, $S_n > S_k$.

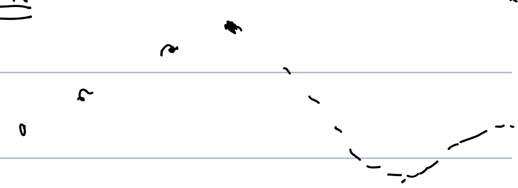
- Consider all dominant terms.

$S_{n_1}, S_{n_2}, \cdots,$

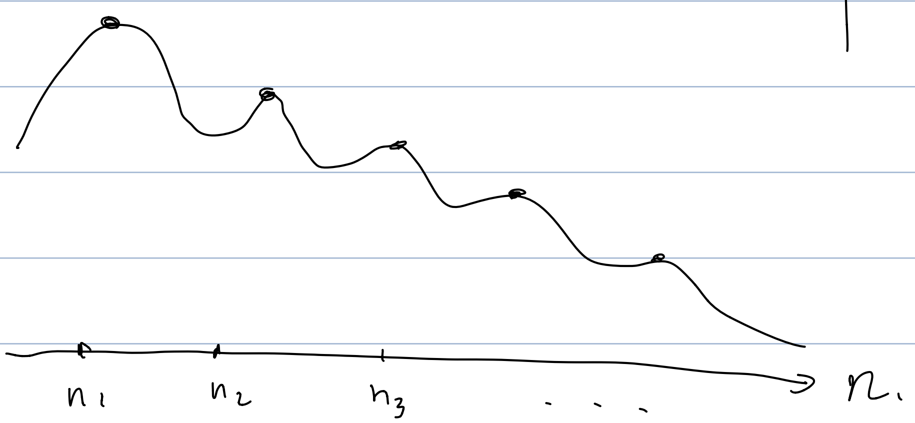
idea: try to construct an increasing sub seq.

- pick S_{n_1}. then, pick $n_2 > n_1$ such that $S_{n_2} \geq S_{n_1}, \cdots$
- trouble: at certain

point, say after getting S_{n_k}, all $n > n_k$, $S_n < S_{n_k}$.



case 1: there are infinitely many such dominant terms.



then $S_{n_1}, S_{n_2}, \cdots$ is a decreasing subsequence.

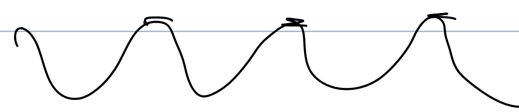
case 2: there are only finitely many such dominant terms $S_{n_1}, \cdots, S_{n_k}$. So, we can consider $(S_n)_{n > n_k}$, try to build an increasing subseq.

By construction, we have.

$\forall\, n > n_k$, $\exists\, m > n$, s.t. $S_m \geq S_n$. (☆)

Hence we can build an increasing subseq,

$S_{m_1}, S_{m_2}, \cdots$ #.

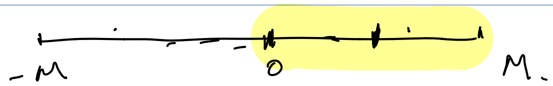


Thm: (Bolzano - Weierstrass Thm).

Every bounded sequence has a convergent subsequence.

Pf: By previous lemma, we have a sub seq that's monotone. Since monotone bounded seq are convergent, we are done. #.

Pf 2: Let's say (s_n) is bounded, $\exists M > 0$, $|s_n| < M$, $\forall n \in \mathbb{N}$.



Let $A_1 = \{ n \in \mathbb{N} \mid s_n \in [0, M] \}$, $B_1 = \{ n \in \mathbb{N} \mid s_n \in [-M, 0] \}$

- since $A_1 \cup B_1 = \mathbb{N}$ an infinite set, hence at least one of A_1, B_1 is infinite. Say A_1 is infinite.

- Cut $[0, M]$ into two halves, and repeat the argument, then $[0, M/2]$ and $[M/2, M]$, at least one of them contains infinitely many points of the sequences.

Then, we get a nested sequence of closed intervals,

$I_1 \supset I_2 \supset I_3 \supset \cdots$, $|I_{n+1}| = \frac{1}{2} |I_n|$.

One can pick sub sequence s_{n_k}, s.t.

$\forall k$. $\underline{s_{n_k} \in I_k}$, and, $\forall k$, $n_{k+1} > n_k$.

Then this subseq. is Cauchy., hence is convergent.

#.

Def: Let (s_n) be a seq in $\mathbb{R}$, A <u>sub sequential limit</u> is any real number or $+\infty$, $-\infty$., that is the limit of a sub sequence of s_n.

Ex: (1) $s_n = (-1)^n \cdot n$.

$+\infty$ and $-\infty$ are subseq. limit.

(2). Consider the enumeration of $\mathbb{Q}$, (r_n), then any real number and $+\infty$, $-\infty$ will occur as subseq limit.

Enumeration of $\mathbb{Q}$:

$0, 1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}, \frac{1}{4}, \frac{3}{4}, -\frac{1}{4}, -\frac{3}{4}, \ldots$

$t \in \mathbb{R}$ is subseq limit of (r_n). $\Leftrightarrow$ $\forall \varepsilon > 0$, $(t-\varepsilon, t+\varepsilon)$ contains infinitely many terms in (r_n).

- Let (s_n) be any seq. Let $\underline{\underline{S}} \subset \mathbb{R} \cup \{+\infty, -\infty\}$, to consist of all subseq limit of (s_n).

Lemma: Let (s_n) be any seq. Then there exists a monotone subseq whose limit is $\limsup s_n$.
(- - - - - subseq $\to \liminf (s_n)$.)

Pf: • If $\limsup s_n = +\infty$, i.e. the original sequence has no upperbound. Then, for each $\overbrace{m \geq 1}^{\text{integer}}$, we can construct s_{n_m}, such that $s_{n_m} > m$, and $n_m > n_{m-1}$.

- If $\limsup s_n = -\infty$, then s_n is unbounded from below, and one can construct a decreasing seq in same way.

- If $\limsup s_n = \alpha \in \mathbb{R}$, then. we can show, $\forall \varepsilon > 0$,

$(\alpha-\varepsilon, \alpha+\varepsilon)$ contains infinitely many terms of the seq. Hence α is a subseq. limit. #.

Thm. (Ross 11.8). (s_n) seq. S = set of subseq limits.

(1). S is non-empty

(2). $\sup S = \limsup s_n$, $\inf S = \liminf s_n$.

(3). $S = \{\alpha\} \iff \lim s_n$ exists, equals $= \alpha$.

Pf: (1) S contains $\liminf s_n$, and $\limsup s_n$.

(2). we are going to prove that

$$(\ast) \quad \liminf s_n \leq \inf S \leq \sup S \leq \limsup s_n.$$

$\forall t \in S$, $\exists$ subseq $(s_{n_k})_k$ of (s_n), that converge to t.

$$\liminf_n (s_n) \leq \liminf_k s_{n_k} \leq \limsup_k (s_{n_k})_k \leq \limsup_n s_n.$$

By convergence of s_{n_k}, we have

$$\liminf_n (s_n) \leq t \leq \limsup s_n$$

Hence, $\limsup (s_n)$ is an upper bound of S, and $\liminf s_n$ —— lower bound of S. Hence $\ast$ is true.

But $\limsup s_n \in S$, $\liminf s_n \in S$. Hence, we get equality $\limsup s_n = \sup S$, $\liminf s_n = \inf S$.

Let (s_n) be a seq. $S = \{$ subseq limit of (s_n).

Thm: S is a closed set.

i.e. $\forall$ sequence in $S \cap \mathbb{R}$. i.e. $(t_n)_n$ $(t_n \in S \cap \mathbb{R})$.

if $\lim t_n = t$ exists in $\mathbb{R} \cup \{+\infty, -\infty\}$, then the limit t also belongs to S.

Pf: Suppose t is finite, we want to show that t is also a subseq. limit of (S_n). Suffice to show $\forall \varepsilon > 0$, $(t-\varepsilon, t+\varepsilon)$ contains infinitely many terms in (S_n). By convergence of $t_n \to t$, $\exists\, t_n$, s.t.

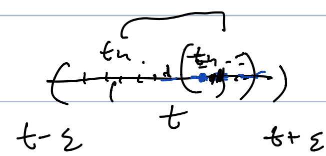
$$|t_n - t| < \frac{\varepsilon}{3}.$$

$t_n \to t$.

Since $t_n \in S$, $\exists\ (S_{n_k})_k \to t_n$,

hence $\exists N > 0$, s.t. $\forall k > N$,

$$|S_{n_k} - t_n| < \frac{\varepsilon}{3}.$$



Thus.
$$\begin{aligned} |S_{n_k} - t| &\le |S_{n_k} - t_n| + |t_n - t| \\ &= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &< \varepsilon. \end{aligned}$$

$\forall k > N$.

$\Rightarrow$ $(t-\varepsilon, t+\varepsilon)$ contains infinitely many terms in (S_n).

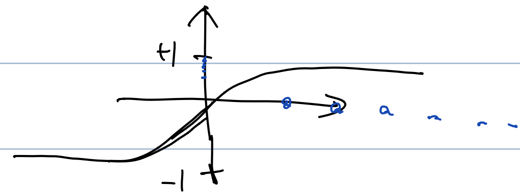
if t is infinite, exercise. #

$(+\infty, -\infty)$. how to think of them?

Consider a monotone bijection between $\mathbb{R}$ and $(-1, +1)$. e.g.

$f : \mathbb{R} \to (-1, +1)$. $\quad f(x) = \dfrac{x}{\sqrt{1+x^2}}$

$\left(\text{or } f(x) = \dfrac{\arctan(x)}{\pi/2} \right)$



$(s_n) \to +\infty \iff f(s_n) \to +1$.

$(s_n) \to -\infty \iff f(s_n) \to -1$.

Hence $\quad \mathbb{R} \cup \{+\infty, -\infty\} \longleftrightarrow (-1, +1) \cup \{+1, -1\}$.

Using this transformation, we can always talk about bounded seq.