Plan for this course: follow Ross

1. Limit, convergence of numbers. (Ch 2). midterm 1.
2. Function. continuous function. (Ch 3).
3. Limit, convergence of functions (Ch 4)

$\left.\begin{array}{l} \text{2.} \\ \text{3.} \end{array}\right\}$ midterm 2.

4. Differentiation & Integrals.

Today: What is the real number $\mathbb{R}$, why we need it?

$\mathbb{N}$: natural number. $= \{1, 2, 3, \cdots\}$.

- supports operation: addition, multiplications.
- "successor", the successor of n is $n+1$.
  - ⇝ chain like structure $1 \to 2 \to 3 \to 4 \to \cdots$.

⇝ mathematical induction:

- statement / proposition, depending on n, call it P_n.
- want to show P_n is true for all $n \in \mathbb{N}$.
- can do it by $\begin{cases} \text{show it for } P_1. & \text{(base step)} \\ \text{show that } P_n \text{ holds} \Rightarrow P_{n+1} \text{ hold.} & \text{(induction step)} \end{cases}$

Ex (from Ross book).

- $|\sin(nx)| \leq n \cdot |\sin x|$. $\forall$ (for all) $n \in \mathbb{N}$, and $x \in \mathbb{R}$.
- $5^n - 4n - 1$ are divisible by 16.

see the proof from the book.

- $\mathbb{Z}$: integers. $\cdots -2, -1, 0, 1, \cdots$
  - has subtraction,
  - 0 is distinguished. $0 + n = n$.

($\mathbb{Z}$ is an example of "ring").

$n, m \in \mathbb{Z}$,

- $\mathbb{Q}$: rational number. $\left(\frac{n}{m}, \quad m \neq 0 \right)$
  - has division operation.
- $\mathbb{Q}$ support : $\cdot$, $/$, $+$, $-$. & compatibility condition.

e.g. $(a+b) \cdot c = ac + bc$.

$\mathbb{Q}$ is a <u>field</u>. (see definition of field. §3). $ab = b \cdot a$.

- $\mathbb{Q}$ has "ordering", i.e. $\leq$. (read about "ordered field" in Ross).

claim:

- $\mathbb{Q}$ has "defects": "$\sqrt{2}$ is not in $\mathbb{Q}$."

$\sqrt{2}$ is a root of $x^2 - 2 = 0$.

$\left(r = \frac{c}{d}, \quad \gcd(c, d) = 1 \right)$

<u>Proposition</u>: If r is a non-zero rational number, and is a root of the following integer coefficient polynomial.

$$C_n \cdot x^n + C_{n-1} \cdot x^{n-1} + \cdots + C_0 = 0.$$

$(C_0 \neq 0, \ C_n \neq 0.)$

$r = \frac{c}{d}$.

then. $\begin{cases} d \text{ divide } C_n. \\ c \text{ divide } C_0. \end{cases}$ ✓

<u>Pf</u> : Plug in $r = \frac{c}{d}$ to the equation,

$$C_n \left(\frac{c}{d} \right)^n + C_{n-1} \left(\frac{c}{d} \right)^{n-1} + \cdots + C_0 = 0$$

multiply by d^n. on both sides. we get

$$\underline{C_n \cdot c^n} + C_{n-1} \cdot c^{n-1} \cdot d + C_{n-2} \cdot c^{n-2} \cdot d^2 + \cdots + C_0 \cdot d^n = 0.$$

(1) put $C_0 \cdot d^n$ on one side,

$$\underline{C_0 \cdot d^n} = -\left[C_n \cdot c^n + C_{n-1} \cdot c^{n-1} \cdot d + \cdots + C_1 \cdot c^1 \cdot d^{n-1} \right]$$

$$= -c \left[C_n \cdot c^{n-1} + C_{n-1} c^{n-2} \cdot d + \cdots + C_1 \cdot d^{n-1} \right].$$

- if both side equal to 0, then $C_0 \cdot d^n = 0 \Rightarrow \begin{matrix} C_0 = 0 \\ \text{or } d = 0 \end{matrix}$
  but this is not allowed.

- thus $c \mid C_0 \cdot d^n$, since c and d are co-prime.
  $\Rightarrow c \mid C_0$.

(2). put $C_n \cdot c^n$ on the other side,

$$C_n \cdot c^n = -d\,\underbrace{[\cdots\cdots]}_{\text{an integer}}.$$

by similar arguments $\left.\begin{matrix} d \mid C_n \cdot c^n, \\ \gcd(d, c) = 1. \end{matrix}\right\} \Rightarrow d \mid C_n$.

#.

Cor: if $r = \frac{c}{d} \neq 0$ is a root of a "monic polynomial",
i.e. lead term has coeff 1.

$$x^n + C_{n-1} x^{n-1} + \cdots + C_0 = 0.$$

then r is an integer.

Pf: since $d \mid C_n = 1$, hence $d = 1$. #.

Apply this to $x^2 - 2 = 0$. If $r = \sqrt{2}$ is a rational number, then $\sqrt{2}$ should be an integer. contradiction.

[if we add in all the roots (in $\mathbb{C} = \{a+ib \mid a, b \in \mathbb{R}\}$) for $\mathbb{Q}$, we still do not get $\mathbb{R}$, but the "algebraic closure". $\overline{\mathbb{Q}} \supset \mathbb{Q}$.
In other word, $\exists\, x \in \mathbb{R}$, that does not satisfy any integer coefficient polynomial equations. e.g. π, e.
]

- $\mathbb{R} \supset \mathbb{Q}$, completeness axioms (§4). next time.