Today : Sequence and Limit.

1. Definitions :

- **sequence** : $a_1, a_2, a_3, \cdots\cdots$ $\qquad a_n \in \mathbb{R}$.

  denote a sequence as $(a_n)_{n\in\mathbb{N}}$.

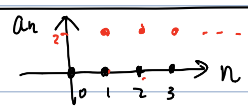
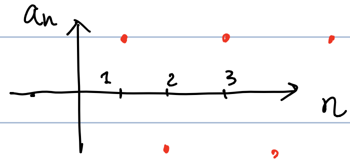
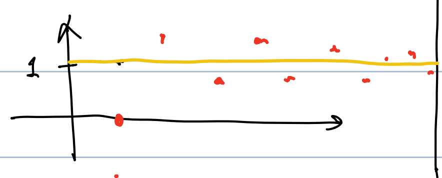
Rmk. : sequence is not a set , $\{a_n\}_{n\in\mathbb{N}} \neq (a_n)_{n\in\mathbb{N}}$.
sequence has its element come in order. (first , second, ---)
set is just information about "Who is in the set".

- **limit** : We say a sequence $(a_n)_n$ has limit $\alpha \in \mathbb{R}$, if
  $\forall \varepsilon > 0$, there exist a real number $N > 0$, such that
  integer
  for all $n > N$, we have.

$$|a_n - \alpha| < \varepsilon.$$

We denote this by $\lim\limits_{n\to\infty} a_n = \alpha$. (or in short $\lim a_n = \alpha$)

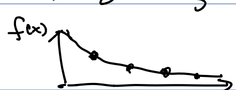
2.

(a_n)	picture	limit (if exists).
• a_1, a_2 $2, 2, 2, \cdots$		2.
• $a_n = (-1)^{n-1}$ $1, -1, 1, -1$.		no limit (oscillation).
• $a_n = 1 + \frac{1}{n}(-1)^n$ $1-1,\ 1+\frac{1}{2},\ 1-\frac{1}{3}, \cdots$		limit = 1.

(§7, §8) Prove by hand, the limit of a sequence is ....

3. Ex:

$$\lim_{n\to\infty} \frac{1}{n^2} = 0 \qquad 1, \frac{1}{2^2}, \frac{1}{3^2}, \cdots\cdots$$

graph.  $f(x) = \frac{1}{x^2}$.

idea:

$\forall \varepsilon > 0$, we want have $\left|\frac{1}{n^2} - 0\right| < \varepsilon$.

$\Leftrightarrow \quad \frac{1}{n^2} < \varepsilon \quad \Leftrightarrow \quad \frac{1}{\varepsilon} < n^2 \quad \Leftrightarrow \quad \frac{1}{\sqrt{\varepsilon}} < n.$

so, we can just take $N = \frac{1}{\sqrt{\varepsilon}}$. Indeed, $\forall n > N$,

we have $n > \frac{1}{\sqrt{\varepsilon}} \Rightarrow n^2 > \frac{1}{\varepsilon} \Rightarrow \varepsilon > \frac{1}{n^2} \Rightarrow \varepsilon > \left|\frac{1}{n^2} - 0\right|$.

(i.e. we met the challenge of showing the sequence eventually falls within ε- distance to $\underline{0}$).

Pf: $\forall \varepsilon > 0$, we want have $\left|\frac{1}{n^2} - 0\right| < \varepsilon$.

$\Leftrightarrow \quad \frac{1}{n^2} < \varepsilon \quad \Leftrightarrow \quad \frac{1}{\varepsilon} < n^2 \quad \Leftrightarrow \quad \frac{1}{\sqrt{\varepsilon}} < n.$

So take $N = \frac{1}{\sqrt{\varepsilon}}$ is enough.

Ex: $\lim\limits_{n\to\infty} \frac{3n+1}{7n-4} = \frac{3}{7}$.

$\frac{4}{3}, \quad \frac{6+1}{10} = \frac{10}{17}\ \text{(sic)}, \quad \frac{3\cdot 3+1}{7\cdot 3-4}, \quad \cdots$

idea: $\left.\begin{array}{l} 3n+1 \approx 3n \\ 7n-4 \approx 7n \end{array}\right\}$ for n large.

hence $\frac{3n+1}{7n-4} \approx \frac{3n}{7n} = \frac{3}{7}$.

Pf:

$\forall \varepsilon > 0$, we want n to be large enough, s.t.

$$\left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < \varepsilon.$$

$$\frac{3n+1}{7n-4} - \frac{3}{7} = \frac{(3n+1)\cdot 7 - 3(7n-4)}{(7n-4)\cdot 7} = \frac{21n+7-(21n-12)}{(7n-4)\cdot 7}$$

$$= \frac{19}{(7n-4)\cdot 7}$$

$\Leftrightarrow \quad \left| \frac{19}{(7n-4)\cdot 7} \right| < \varepsilon. \qquad \because \ 19>0, \quad 7n-4>0$ for $n \in \mathbb{N}$.

$\Leftrightarrow \quad \frac{19}{(7n-4)\cdot 7} < \varepsilon \quad \Leftrightarrow \quad \frac{19}{7\varepsilon} < 7n-4.$

$\Leftrightarrow \quad \frac{19}{7\varepsilon} + 4 < 7n \quad \Leftrightarrow \quad n > \underbrace{\frac{1}{7}\left(\frac{19}{7\varepsilon}+4\right)}_{N}.$

So, if we take $N = \frac{1}{7}\left(\frac{19}{7\varepsilon}+4\right)$, then $n > N$

$\Rightarrow \left| \frac{3n+1}{7n-4} - \frac{3}{7} \right| < \varepsilon.$

(actually $\Leftrightarrow$.)

Ex: $\lim_{n\to\infty} 1 + \frac{1}{n}(-1)^n = 1.$

Pf: $\forall \varepsilon > 0$, we want n large enough, s.t.

$|a_n - 1| < \varepsilon. \Leftrightarrow \left| 1 + \frac{1}{n}(-1)^n - 1 \right| < \varepsilon$

$\Leftrightarrow \left| \frac{1}{n}(-1)^n \right| < \varepsilon.$

$\Leftrightarrow \frac{1}{n} < \varepsilon \quad \Leftrightarrow \quad \underline{n > \frac{1}{\varepsilon}}.$

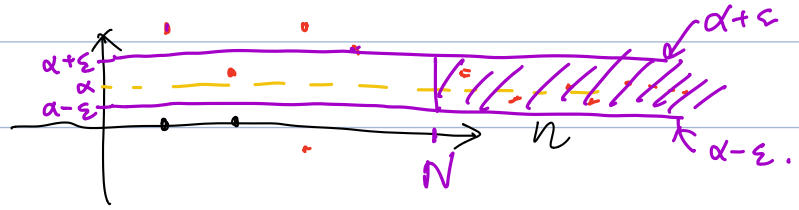
Just take $N = \frac{1}{\varepsilon}$, then $n > N \Rightarrow |a_n - 1| < \varepsilon.$ #

§9. Property and tools to find limit.

- Bounded sequence: $a_1, a_2, \cdots$ is a bounded sequence, if $\exists M > 0$, such $-M \le a_n \le M$ for all $n \in \mathbb{N}$.

- **Thm:** all convergent sequence are bounded.

idea:

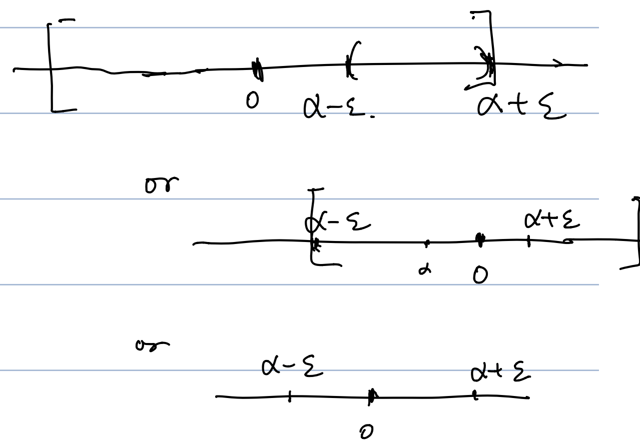


$\forall \varepsilon > 0$, $\exists N$, s.t. $|a_n - \alpha| < \varepsilon$ $\forall n > N$

Say $\lim a_n = \alpha$.

Pf: Fix an $\varepsilon > 0$, then by convergence of the sequence $\exists N$, s.t. $\forall n > N$, $|a_n - \alpha| < \varepsilon$. $\Leftrightarrow$ $\alpha - \varepsilon < a_n < \alpha + \varepsilon$

Let $C_1 = \max(|\alpha+\varepsilon|, |\alpha-\varepsilon|)$.



then $\alpha + \varepsilon \leq |\alpha+\varepsilon| \leq C_1$

$\alpha - \varepsilon \geq -|\alpha - \varepsilon| \geq -C_1$

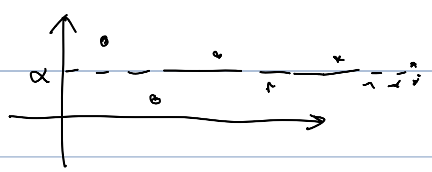
then $a_n \in (-C_1, C_1)$.

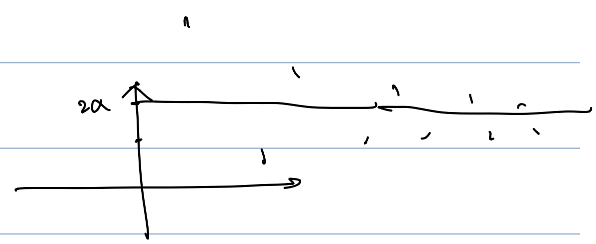
Let n_0 be the largest integer $\leq N$.

Then. Let $M = \max(|a_1|, \cdots, |a_{n_0}|, C_1)$, we have $\forall n \in \mathbb{N}$, $|a_n| \leq M$. Thus, the sequence is bounded.

- **Thm:** If $\lim a_n = \alpha$, and if $k \in \mathbb{R}$. then $\lim (k \cdot a_n) = k \cdot \alpha$.

Idea:





Pf: $\forall \varepsilon > 0$, need to find N, s.t. $\forall n > N$,

$|k a_n - k\alpha| < \varepsilon$.

If $k = 0$, then automatically true. can take $N = 1$.

If $k \neq 0$, then $|k| \neq 0$. Then.

$$\Leftrightarrow \quad |k| \cdot |a_n - \alpha| < \varepsilon$$

$$\Leftrightarrow \quad |a_n - \alpha| < \frac{\varepsilon}{|k|}.$$

- By convergence of a_n to α, if we set $\varepsilon' = \varepsilon/|k|$, then $\exists N$, s.t. $\forall n > N$, $|a_n - \alpha| < \varepsilon' = \varepsilon/|k|$. This N satisfies our need.

- <u>Thm</u>: Let a_n, b_n be ~~~~ convergent sequences. $\lim a_n = \alpha$, $\lim b_n = \beta$.

Then ①. $\lim (a_n + b_n) = (\lim a_n) + (\lim b_n) = \alpha + \beta$.

② $\lim (a_n \cdot b_n) = (\lim a_n) \cdot (\lim b_n) = \alpha \cdot \beta$.

③ if $a_n \neq 0$ $\forall n$. and if $\alpha \neq 0$. then.

$\lim (1/a_n) = \frac{1}{\alpha}$.

<u>Pf</u>: ①. $\forall \varepsilon > 0$, ~~~~ wait. n large enough, s.t.

$$|a_n + b_n - (\alpha + \beta)| < \varepsilon$$

$$\Leftrightarrow \quad |(a_n - \alpha) + (b_n - \beta)| < \varepsilon.$$

$$\Leftarrow \quad |a_n - \alpha| + |b_n - \beta| < \varepsilon.$$

$$\Leftarrow \quad \underline{|a_n - \alpha| < \tfrac{\varepsilon}{2}}, \text{ and } \underline{|b_n - \beta| < \tfrac{\varepsilon}{2}}.$$

$|A + B| \leq |A| + |B|$.

we can let n be large enough, to achieve both conditions.

② $\forall \varepsilon > 0$, wait to show, if n is large enough, then.

$$|a_n b_n - \alpha \cdot \beta| < \varepsilon.$$

$a_n = \alpha + \underline{(a_n - \alpha)}$, $\quad b_n = \beta + (b_n - \beta)$.

$$a_n \cdot b_n = [\alpha + (a_n - \alpha)] \cdot [\beta + (b_n - \beta)].$$

$$= \alpha \cdot \beta + (a_n - \alpha) \cdot \beta + \alpha \cdot (b_n - \beta) + (a_n - \alpha)(b_n - \beta).$$

$$a_n \cdot b_n - \alpha\beta = (a_n-\alpha)\cdot\beta + \alpha\cdot(b_n-\beta) + (a_n-\alpha)\cdot(b_n-\beta).$$

$\forall \delta > 0$, $\exists N_\delta$; such that $|a_n-\alpha| < \delta$, $|b_n-\beta| < \delta$, $\forall n > N_\delta$.

thus. $|a_n b_n - \alpha\beta| \le |a_n-\alpha|\cdot|\beta| + |\alpha|\cdot|b_n-\beta| + |a_n-\alpha|\cdot|b_n-\beta|$

$$\le \delta\cdot|\beta| + |\alpha|\cdot\delta + \delta^2.$$

take $\delta < 1$. , then $\delta^2 < \delta$. $\le \delta(1+|\alpha|+|\beta|)$.

given.

Now, we choose δ small enough. s.t. $\delta(1+|\alpha|+|\beta|) < \varepsilon$

take $\delta = \frac{1}{1+|\alpha|+|\beta|}\cdot\varepsilon$. (or if it ≥ 1, take $\delta = 1$). and let

$N = N_\delta$. , then $\forall n > N$,

$$|a_n b_n - \alpha\beta| < \delta(1+|\alpha|+|\beta|) \le \varepsilon.$$

(3). $\forall \varepsilon > 0$, want to show. if n large enough.

(next time. revisit).

$$\left|\frac{1}{a_n} - \frac{1}{\alpha}\right| < \varepsilon.$$

$$\Leftrightarrow \left|\frac{\alpha - a_n}{a_n\cdot\alpha}\right| < \varepsilon$$

need a Lemma, showing. $\left|\frac{1}{a_n}\right|$ is bounded above.

i.e. a_n is not arbitrarily close to 0. Assume we have this claim, that. $\exists C > 0$, $\left|\frac{1}{a_n}\right| < C$, $\forall n \in \mathbb{N}$, then.

$$\Leftrightarrow \left|\frac{\alpha - a_n}{a_n\,\alpha}\right| < \varepsilon \Leftarrow \left|\frac{\alpha-a_n}{\alpha}\right|\cdot C < \varepsilon.$$

$$\Leftrightarrow |\alpha - a_n| < \frac{\varepsilon}{C}\cdot|\alpha|.$$

This can be met by convergence of a_n to α.