(Rudin Ch 2. on connectedness.)

Recall:

- a topological space : a set X, together with a collection of subsets of X, called open subsets. such that
  - X, ϕ are open
  - $\bigcup_{\alpha \in A} U_\alpha$ is open, A avibitrary index set., U_α open.
  - $\bigcap_{i=1}^{N} U_i$ is open. (finite intersection).

- Given a top. space X, and $S \subset X$, we endow S with the induced topology:

  $U \subset S$ is open in S iff $\exists \tilde{U} \subset X$, open in X, s.t. $U = \tilde{U} \cap S$.

Ex: $X = \mathbb{R}$, $S = \mathbb{Z} \subset \mathbb{R}$. with induced topology on S,

$\forall n \in \mathbb{Z}$, $\{n\}$ is open in S Pf: $\underset{U}{\{n\}} = \underbrace{(n-\frac{1}{2}, n+\frac{1}{2})}_{\tilde{U},\ \text{open in } X.} \cap S$

Ex: $X = \mathbb{R}$, $S = [0, 1)$. Then S is open in S.

S is not open in X.

Cor: (a) If $S \subset X$ is open in X, then $U \subset S$ is open in S iff U is open in X.

(b) If $S \subset X$ is closed in X, then $E \subset S$ is closed in S iff E is closed in X.

Ex. · $X = \mathbb{R}$, $S = (0,1)$. $U = (\frac{1}{3}, \frac{1}{2}) \subset S$. U is open in S.

U is open in X.

· $X = \mathbb{R}$, $S = [0,1]$. $U = [0, \frac{1}{2}] \subset S$. U is closed in S

and U is closed in X.

Pf: (a) $U \subset S$ open in S $\Rightarrow$ $U \subset S$ open in X.

By def, $U = \tilde{U} \cap S$, for some $\tilde{U} \subset X$ open in X.

since $\tilde{U}$ and S are both open in X, hence U is open in X.

$\Leftarrow$ if U is open in X, then

$$U = U \cap S$$

Hence U is open in S.

(b) similar.

Recall: · A top. space X is connected if the only subsets of X that is both open and closed are X and $\emptyset$.

· X is not connected iff $X = \underbrace{U \amalg V}_{\text{disjoint union}}$

U, V are non-empty open subset of X.

Def: Let X be a top. space. $E \subset X$ $\overbrace{\text{subset with induced topology}}^{\text{"subspace"}}$

E is a connected subset iff E as a top. space is connected.

Lemma: E is connected iff E cannot be written as $A \cup B$.

(equivalence of definition with Rudin). where $\bar{A} \cap B = \emptyset$, and $A \cap \bar{B} = \emptyset$.

(here, closure are taken w.r.t. ambient space X.)

Pf: $\Rightarrow$. ~~By co~~ Prove by contradiction. Suppose $E = A \cup B$, $\bar{A} \cap B = \emptyset$, $A \cap \bar{B} = \emptyset$,

then. $A = \bar{A} \cap E$, hence A is closed in E.

($\because$ $\bar{A} \cap E = \bar{A} \cap (A \cup B) = (\bar{A} \cap A) \cup (\bar{A} \cup B) = A \cup \emptyset = A$).

Similarly, B is closed in E. $\because$ $A = E \setminus B$, $\therefore$ A is also open.

Hence, E is not connected. contradiction.

$\Leftarrow$ as exercise.

Thm (2.47) Rudin $E \subset \mathbb{R}$ is connected

$\Leftrightarrow$ $\forall\, x, y \in E$, $x < y$. we have $[x,y] \subset E$.

Pf: $\Rightarrow$ Prove by contradiction. Suppose $\exists\, x, y \in E$, $x < y$, such $[x,y] \not\subset E$. This means, $\exists\, z \in [x,y]$, s.t. $z \notin E$. Since $x, y \in E$. $z \in (x,y)$. Let $A = E \cap (-\infty, z)$, $B = E \cap (z, +\infty)$.

Then, A, B are open in E., and non-empty. ($x \in A$, $y \in B$).

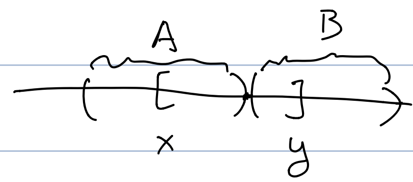
and $E = A \amalg B$. Hence E is not connected. Contradiction!

$\Leftarrow$ Prove by contradiction. Suppose $E = A \amalg B$, A, B non-empty, disjoint open subsets in E. Pick $x \in A$, and $y \in B$.

WLOG, assume $x < y$. Then, by hypothesis, $[x,y] \subset E$.

Let $\tilde{A} = [x,y] \cap A$,

$\tilde{B} = [x,y] \cap B$.



Since A, B are both open and closed in E.

Hence $\tilde{A}, \tilde{B}$ are both open and closed in $[x,y]$.

Since $[x,y]$ is closed in $\mathbb{R}$, $\tilde{A}, \tilde{B}$ are also closed in $\mathbb{R}$.

Let $z = \sup(\tilde{A})$. Since $\tilde{A}$ is closed and bounded in $\mathbb{R}$, $z \in \tilde{A}$. $z \neq y$, since $y \in \tilde{B}$, $y \notin \tilde{A}$. Hence $(z, y] \subset \tilde{B}$.

Since $\tilde{B}$ is closed in $\mathbb{R}$, $\therefore \overline{(z,y]} \subset \tilde{B}$ $\therefore z \in [z,y] = \overline{(z,y]} \subset \tilde{B}$.

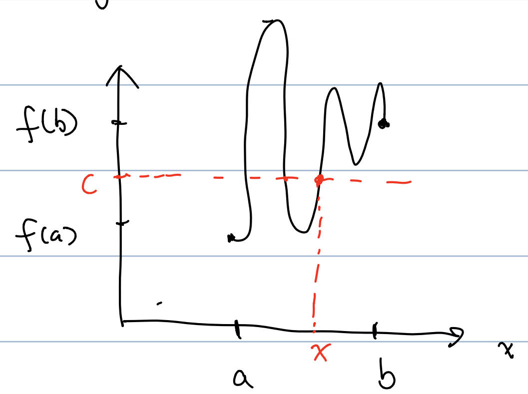
Thus, $z \in \tilde{A}$ and $z \in \tilde{B}$. contradiction with $\tilde{A} \cap \tilde{B} = \phi$. #.

Return to Ch 4. Continuous fcn and connectedness.

Recall: If $f: X \to Y$ is cont., $E \subset X$ is connected. then $f(E)$ in Y is connected.

In the special case, $f: \mathbb{R} \to \mathbb{R}$ continuous.

Thm: If $f: [a,b] \to \mathbb{R}$ is cont. and if $f(a) < f(b)$, and c is such that $f(a) < c < f(b)$, Then $\exists\ x \in (a,b)$. s.t. $f(x) = c$.



Pf: • Since $[a,b] \subset \mathbb{R}$ is connected, we have $f([a,b]) \subset \mathbb{R}$ is connected.

• Since $f(a)$, $f(b) \in f([a,b])$, hence by Thm 2.47.

$$[f(a), f(b)] \subset f([a,b]).$$

In particular, if $c \in (f(a), f(b))$, then

$$c \in f([a,b]).$$

Let $x \in [a,b]$, s.t. $f(x) = c$. Then $x \neq a, b$.

Hence $x \in (a,b)$. #.

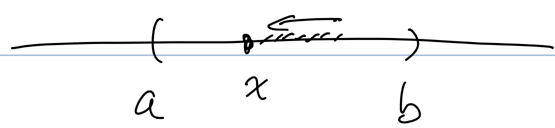
Discontinuity for real valued functions:

Recall: $f: X \to Y$ is discontinuous at $x \in X$.

if and only if x is a limit point of X., and $\lim_{q \to x} f(q)$ ~~does~~ either doesn't exist or. does not equal to $f(x)$.

Def: Let $f: (a,b) \to \mathbb{R}$. (not necesarily continuous).

- $\forall x \in [a,b)$, we say $\boxed{f(x+) = q}$. if for all seq (t_n) in (x,b) that converge to x, we have limit $\lim_n f(t_n) = q$.



- $\forall x \in (a,b]$, we say $f(x-) = q$. if for all seq (t_n) in (a,x), converging to x, we have $\lim_{n \to \infty} f(t_n) = q$

Equivalently, one can use ε-δ language. For example,

$f(x+) = q$ iff $\forall \varepsilon > 0,\ \exists \delta > 0$. s.t. $\forall\ t \in (x-\delta, x) \cap (a,b)$

$|f(t) - q| < \varepsilon$.

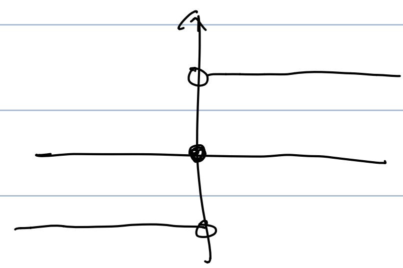
- Say $f(x+)$: the right limit of f at x.
  $f(x-)$ ---- left ———— — —.

<u>Def</u>: Say $f: (a,b) \to \mathbb{R}$. $x_0 \in (a,b)$. Suppose f is not continuous at x_0.

(1st kind discon...)

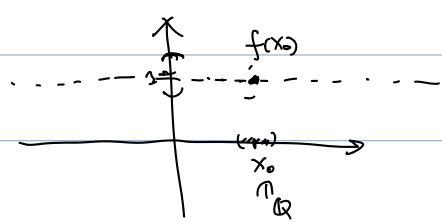
- we say f has <u>simple discontinuity</u> at x_0, if both $f(x_0+)$ and $f(x_0-)$ exists. (i.e. either $f(x_0+) \neq f(x)$ or $f(x_0-) \neq f(x)$).
- otherwise we say f has <u>2nd kind of discontinuity</u>.

<u>Ex</u>: (a). $f(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$



discontinity of 1st kind (simple) discontimity.

(b). $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$



- If $\boxed{x_0 \in \mathbb{Q},}$ is $f(x)$ continuous at x_0 ?

Ans: Not continuous. Use ε-δ languge. $\exists\, \varepsilon$, say $\varepsilon = \frac{1}{2}$,

such $\forall\, \delta > 0$, $f((x_0-\delta, x_0+\delta)) = \{0, 1\} \not\subset B_{\frac{1}{2}}(f(x_0)) = B_{\frac{1}{2}}(1)$.

there are irrationals in $(x_0-\delta, x_0+\delta)$. $\quad = (-\frac{1}{2}, \frac{3}{2})$.

We can also use seq def. to show f is not cont. at x_0.

Let x_n be a seq of irrational, converging to x_0.

then $\lim\limits_{n\to\infty} f(x_n) \neq f(\lim\limits_{n\to\infty} x_n)$, Hence, f is not cont. at x_0.

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0. \qquad f(\lim_{n\to\infty} x_n) = f(x_0) = 1.$$

(2). $$f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

(3) $$f(x) = \begin{cases} \frac{1}{n} & x \in \mathbb{Q},\quad x = \frac{m}{n},\ (m,n) \text{ coprime} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$