• Connectedness :

Recall : • A Topological space X is a set X, together with a collection of subsets of X, called "open subsets", s.t. ① X, ϕ open ② $\bigcup_{arb}$ (open) is open ③ $\bigcap_{finite}$ (open) is open.

induced topology

• For a top. sp. X, $S \subset X$ a subset. we say $U \subset S$ is open (in S) iff $\exists \tilde{U} \subset X$ open in X, s.t. $U = \tilde{U} \cap S$.

• For a top sp X, X is connected if X cannot be written as $X = U \cup V$ where $U, V \neq \phi$, $U \cap V = \phi$, U, V open in X.

Lemma : Let X be a top. sp., $S \subset X$ subset. TFAE.

(1) S is connected, when S is endowed with the induced topology

(Rudin's def'n for connected subset).

(2). S cannot be written as $A \cup B$, where $\bar{A} \cap B = \phi$ and $A \cap \bar{B} = \phi$.

($\overline{(\cdots)}$ means taking closure in X).

Pf : (1) $\Rightarrow$ (2). Prove by contradiction. Suppose $S = A \cup B$ with $\bar{A} \cap B = \phi$, $A \cap \bar{B} = \phi$. Then, A, B are both open and closed in S. Indeed,

$$\bar{A} \cap S = \bar{A} \cap (A \cup B) = (\bar{A} \cap A) \cup (\bar{A} \cup B) = A \cup \phi = A$$

Hence A is closed in S. Similarly, B is closed in S.

Since the complement of A in S is B, which is closed.
Hence, A is open in S. similarly B is open in S.
Hence S is a disjoint union of non-empty open sets.
$\Rightarrow$, S is not connected.

$(2) \Rightarrow (1)$. Suppose S is not connected. then
$S = A \cup B$. A, B non empty, open in S, disjoint.
A, B are also closed in S. Hence, $A = F \cap S$, for some ~~close~~ subset $F \subset X$ closed. In particular, $A = \bar{A} \cap S$.
This implies $\bar{A} \cap B = \phi$. Similarly, $\bar{B} \cap A = \phi$. This contradict with (2). #.

Lemma 2 (induced topology). Let X be a top. space, $S \subset X$ subspace.
("subspace" means subset, with induced topology). Then.
(1) if S is open in X. then $U \subset S$ is open in S iff U is open in X
(2) if S is closed in X, then $U \subset S$ is closed in S iff U is closed in X.

Pf: exercise. #.

Now, we finish the leftover from last time.

Thm: Let $E \subset \mathbb{R}$ be a subset. then
(2.47) in Rudin.
E is connected $\Leftrightarrow$ $\forall x, y \in E$, and $x<y$, we have $[x, y] \subset E$.

Pf: $\Rightarrow$ Suppose $[x,y] \not\subset E$, then $\exists\, z \in [x,y]$, $z \notin E$.

$z \neq x, y$. Hence $z \in (x,y)$. Now

$$E = \big[(-\infty, z) \cap E\big] \cup \big[(z, +\infty) \cap E\big]$$

two open subsets in E, non-empty. (x, y are in them).

E is not connected. A contradiction.

$\Leftarrow$ Suppose E is not connected. then

$E = A \amalg B$, A, B are both open and closed in E.

disjoint union

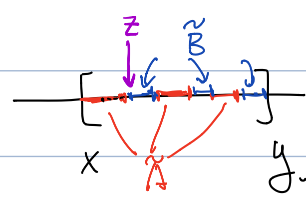
Pick $x \in A$, $y \in B$. WLOG, assume $x < y$. Then by ~~condit~~ hypothesis, $[x,y] \subset E$. Let $\tilde{A} = [x,y] \cap A$, $\tilde{B} = [x,y] \cap B$. Then

$$[x,y] = \tilde{A} \amalg \tilde{B}, \quad \tilde{A}, \tilde{B} \text{ both open and closed in } [x,y].$$

- Since $\tilde{A}, \tilde{B} \subset [x,y]$ are closed in $[x,y]$, and $[x,y] \subset \mathbb{R}$ in $\mathbb{R}$, hence $\tilde{A}, \tilde{B} \subset \mathbb{R}$ are closed in $\mathbb{R}$.

- Let $z = \inf(\tilde{B})$. Since $\tilde{B}$ is bounded, $z \in \mathbb{R}$.

  Since $\tilde{B}$ is closed, $\underline{z \in \tilde{B}}$.



  Hence $z \neq x$, since $x \in \tilde{A}$, $x \notin \tilde{B}$.

  Then, $\exists$ a sequence (t_n) in (x, z), ~~cott~~ converging to z.

  $(t_n) \to z$, $(t_n) \in \tilde{A}$, $\tilde{A}$ is closed in $\mathbb{R}$ $\Rightarrow$ $\underline{z \in \tilde{A}}$.

alternatively, since $[x,z) \subset \tilde{A}$ and $\tilde{A}$ is closed, Hence $[x,z] \subset \tilde{A}$. $\Rightarrow z \in \tilde{A}$

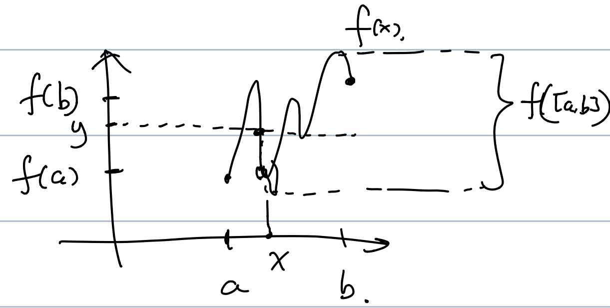
Thus $z \in \tilde{A}$ and $z \in \tilde{B}$. ~~//~~ contradict with $\tilde{A} \cap \tilde{B} = \phi$.

#.

Now, back to Ch 4. (Rudin):

Recall: if $f: X \to Y$. is cont. and X is connected, then $f(X)$ is connected.

Cor: If $f: [a,b] \to \mathbb{R}$ is continuous., and assume $f(a) < f(b)$, then. $\forall y \in (f(a), f(b))$, $\exists x \in (a,b)$, s.t. $f(x) = y$.



Pf: Since $[a,b]$ is connected., ($[a,b]$ satisfies the criterin on RHS of Thm 2.47), then $f([a,b])$ is connected.

Since $f(a), f(b) \in f([a,b])$. by Thm 2.47. we know $[f(a), f(b)] \subset f([a,b])$. #.

(Intermediate value property).

Discontinuity of functions on $\mathbb{R}$ to $\mathbb{R}$.

Recall limit of a function: say $f: E \to Y$, $E \subset X$. any function.

say $x \in X$ is a limit point of E, i.e. ($\forall B_\varepsilon(x) \setminus \{x\}. = \overset{\times}{B}_\varepsilon(x)$ ← cross.

$\overset{\times}{B}_\varepsilon(X) \cap E. \neq \phi.$), then we say $\lim_{t \to x} f(t) = y$, if

$\forall \varepsilon > 0$, $\exists \delta > 0$, s.t.

$$f(\overset{\times}{B}_\delta(x) \cap E). \subset B_\varepsilon(y).$$

Equivalently: for all seq $t_n \to x$, $t_n \neq x$, $t_n \in E$,

we have $\lim_{n\to\infty} f(t_n) = y$

Def: (left and right limit): Let $f: (a,b) \to \mathbb{R}$. $x_0 \in (a,b)$.

- We say $f(x_0+) = y$, if $\lim_{t\to x_0} f|_{(x_0,b)}(t) = y$.

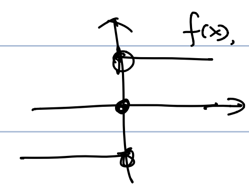
- Similarly, say $f(x_0-) = y$, if $\lim_{t\to x_0} f|_{(a,x_0)}(t) = y$.

Now: f is continuous at x_0, iff.

$$f(x_0) = f(x_0+) = f(x_0-).$$

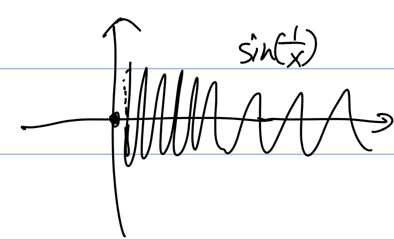
- If f is not continuous at x_0. If $f(x_0+)$ and $f(x_0-)$ exist, we say f has a simple discontinuity at x_0.
  or 1st kind discontinuity.
  Otherwise, ~~then~~ if $f(x_0-)$ or $f(x_0+)$ do not exist, then ...
  2nd kind of discontinuity.

Ex: (1) $f(x) = \begin{cases} -1 & x<0 \\ 0 & x=0 \\ +1 & x>0 \end{cases}$



discont. at $x=0$. 1st kind of discon.

(2). $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x>0 \\ 0. & x\le 0 \end{cases}$



$f(0+)$ doesn't exist. ~~f~~ $\Rightarrow$ f has a
$f(0-) = 0$. 2nd kind disc.

(3). $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$

then, $\forall x \in \mathbb{R}$, f is disc. at x.
2nd kind of disc. $f(x-)$, $f(x+)$ don't exist.
One can approach any $x \in \mathbb{R}$, using purely $\mathbb{Q}$ or $\mathbb{R}\setminus\mathbb{Q}$.

(4). $f(x) = \begin{cases} \frac{1}{n} & x \in \mathbb{Q}, \quad x = \frac{m}{n}, \ m, n \text{ coprime}, \ n > 0. \\ 0 & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$

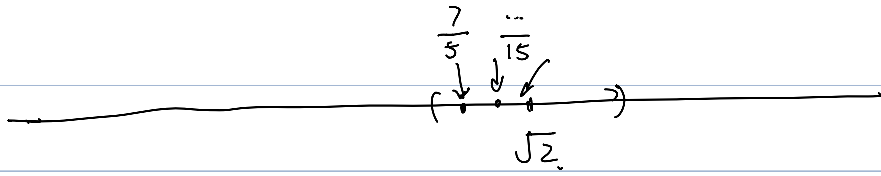
(HW. #18.)
Rudin.

<u>Claim</u>: • $\forall x \in \mathbb{R}\setminus\mathbb{Q}$, f is continuous

• $\forall x \in \mathbb{Q}$, f has simple discontinuity.

$\boxed{\text{Ex}}$. take $x = \sqrt{2}$, let x_n be a sequence, $x_n \to \sqrt{2}$.
Let x_n be purely rational. What is $\lim f(x_n)$?

• Pick δ, look at $B_\delta(\sqrt{2}) \cap \mathbb{Q}$, write each rational in here as $\frac{m}{n}$. Take all those with the smallest denominators, it is a finite set.



- one can get a seq of rationals, (x_n), such that $x_n \to \sqrt{2}$, and denominator$(x_n) \to \infty$. For such x_n, $\lim_{n\to\infty} f(x_n) = 0$.

Claim: $\forall x \in \mathbb{R}$, $f(x+) = f(x-) = 0$.

Monotone functions on $\mathbb{R}$

- $f : (a,b) \to \mathbb{R}$. We say f is (weakly) monotone increasing, if $\forall\, a < x < y < b$, we have $f(x) \le f(y)$.

Similarly define monotone decreasing.

Thm 1: if $f : (a,b) \to \mathbb{R}$ is monotone increasing, then $f(x+)$ and $f(x-)$ exists at every point $x \in (a,b)$. More precisely,

$$f(x-) = \sup \{ f(t) \mid t \in (a, x) \}$$
$$f(x+) = \inf \{ f(t) \mid t \in (x, b) \}.$$

If $x < y$, then
$$f(x+) = \inf \{ f(t) \mid x < t < y \} \le \sup \{ f(t) \mid x < t < y \} = f(y-).$$

Thm 2: $f : (a,b) \to \mathbb{R}$ monotone increasing fcn.

then f has at most countably many discontinuities.

Pf: Let x be a discontinuity of f, define

$$I(x) = (f(x-), f(x+)).$$

Define

$$I = \bigcup_{\substack{x \text{ a} \\ \text{discontinuity} \\ \text{of } f}} I_x.$$

then I is a union of disjoint open subsets.

Hence, by our homework results, I is at most countable disjoint union of open intervals. #.

Discussion:

- If $f: [0,1] \to \mathbb{R}$, cont. and $f([0,1]) \subset [0,1]$. Show that, $\exists x \in [0,1]$. s.t. $f(x) = x$.