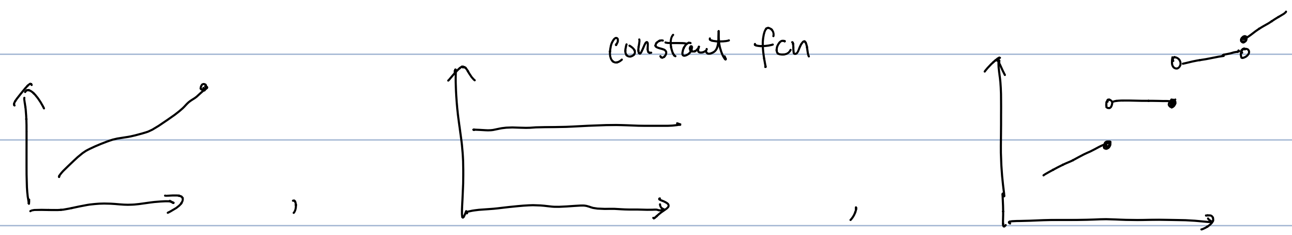
Today:

- monotone functions (Rudin. Ch 4.5).
- sequence and convergence of functions. (Ross Ch 4; Rudin Ch 7, 1, 2.)

(more generally, $f: (a,b) \to \mathbb{R}$).

- **Def:** a function $f: \mathbb{R} \to \mathbb{R}$ is monotone increasing. if $\forall\ x > y$, we have $f(x) \geq f(y)$.

Ex:



(similarly, one can define monotone decreasing).

Thm: Suppose $f: (a,b) \to \mathbb{R}$ is a monotone increasing fcn, then $\forall\ x \in (a,b)$, the left limit $f(x-)$ and the right limit $f(x+)$, exist, they satisfys:

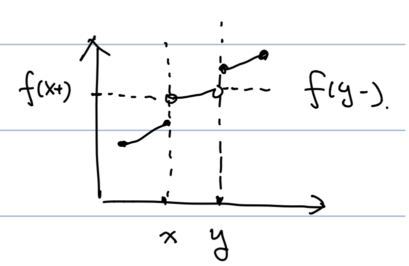
(1) $f(x-) = \sup \{ f(t) \mid t < x \}$.

$f(x+) = \inf \{ f(t) \mid t > x \}$.

$f(x-) \leq f(x+)$.

(2) Given $x < y$ in (a,b), then.

$f(x+) \leq f(y-)$.



If f is monotone, then

Cor: $f(x)$ only has discontinuity of the first kind.

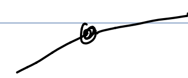
Prop: If f is monotone, there are at most <u>countably</u> many discontinuities.

Rmk: without the monotone condition, we can have function

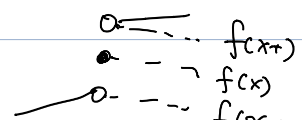
$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

$f(x)$ is discontinuous $\forall x \in \mathbb{R}$. $\mathbb{R}$ is an uncountable set.

- If $\alpha \notin \mathbb{Q}$, then let $x_n \to \alpha$ be a rational number seq. then $\lim_{n\to\infty} f(x_n) \neq f(\lim_{n\to\infty} x_n)$, so f is not cont. at α.
- If $\alpha \in \mathbb{Q}$, then let $x_n \to \alpha$ be a seq of irrationals. then $\lim_{n\to\infty} f(x_n) \neq f(\lim_n x_n)$



Pf: If x is a discontinuity of f, then.



$f(x-) < f(x+)$, and $f(x-) \leq f(x) \leq f(x+)$.

Associate to x a rational number $r_x \in (f(x-), f(x+))$.

Then for each x disc of f, we have $r_x \in \mathbb{Q}$.

If $x < y$, disc of f, then $r_x < f(x+) \leq f(y-) < r_y$,

hence r_x, r_y are distinct.

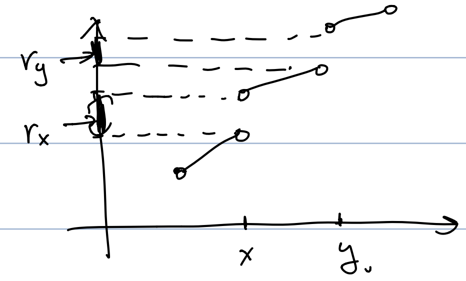
the set of discontinuity of f.

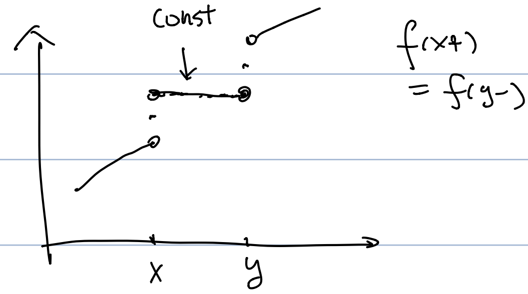
Then, we have an injective map from $\mathrm{Disc}(f) \hookrightarrow \mathbb{Q}$.

Hence $\mathrm{Disc}(f)$ is countable, since it can be ~~inde~~ identified with a subset of countable set.

either finite.

infinite, but has bijection with $\mathbb{N}$.





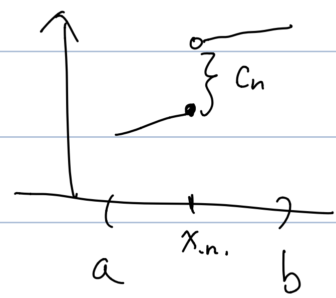
Construction of an example with countably many discon.

① Let (x_n) be a seq of distinct points in (a,b).
e.g. an enumeration of rationals.

② Let c_n be a seq of positive numbers, such that
$\sum c_n < \infty$. (e.g. $c_n = \frac{1}{2^n}$).

③ Let $f(x) = \sum_{\substack{n \in \mathbb{N}, \\ x_n < x}} c_n$, $x \in (a,b)$.

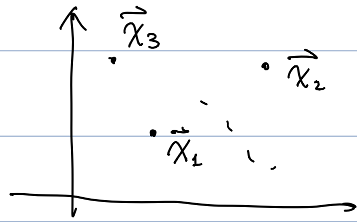
One can check that, $f(x)$ has discontinuity exactly at $\{x_n \mid n \in \mathbb{N}\}$, $f(x_n+) - f(x_n-) = c_n$.
$f(x_n) = f(x_n-)$.



- Sequence and Convergence of function.
  - seq of real numbers, $(x_n)_{n \in \mathbb{N}}$, $x_n \in \mathbb{R}$.

we say $x_n \to x$, if $\forall \varepsilon > 0$, $\exists N > 0$, s.t. $\forall n > N$, $|x_n - x| < \varepsilon$.

• seq of points $(\vec{x}_n)_n$. $\quad \vec{x}_n \in \mathbb{R}^m$



$\vec{x}_n = (x_{n,1}, x_{n,2}, \cdots x_{n,m}) \in \mathbb{R}^m$

$(\vec{x}_n) \longrightarrow \vec{x}$ in the Euclidean metric sense of $\mathbb{R}^m$

$\iff (x_{n,i}) \longrightarrow x_i \quad \forall i \in \{1, \cdots, m\}$. "convergence in each component"

☆ • seq of seq. : let each $\vec{x}_n$ denote a seq

$\vec{x}_n = (x_{n1}, x_{n2}, x_{n3}, \cdots\cdots) \in Map(\mathbb{N}, \mathbb{R}) = \mathbb{R}^{\mathbb{N}}$.

a seq of seq $(\vec{x}_n)_n$:

$\vec{x}_1 = (x_{11}, x_{12}, x_{13}, \cdots\cdots)$

$\vec{x}_2 = (x_{21}, x_{22}, x_{23}, \cdots\cdots)$.

Several possibilities:

$\vec{x} = (x_1, x_2, \cdots\cdots)$

$\vec{y} = (y_1, y_2, \cdots\cdots)$.

• Define a "metric" on $\mathbb{R}^{\mathbb{N}}$,

useful in some Lebesgue integral theory. $\longrightarrow d_2(\vec{x}, \vec{y}) = \left(\sum_{i=1}^{\infty} (x_i - y_i)^2 \right)^{\frac{1}{2}}$.

or.

sup distance is used in uniform convergence. $\rightsquigarrow d_\infty(\vec{x}, \vec{y}) = \sup \{ |x_i - y_i| : i \in \mathbb{N} \}$

$\left(= \lim_{p \to \infty} \left(\sum_{i=1}^{\infty} |x_i - y_i|^p \right)^{\frac{1}{p}} \right)$.

In this case, these different metrics are not "equivalent".

[Recall that, 2 metric functions d and $\tilde{d}$ are equiv. if $\exists C > 0$, such that, $\forall x, y \in X$.

$\frac{1}{C} \tilde{d}(x,y) \le d(x,y) \le C \cdot \tilde{d}(x,y)$]

Def: Let $(\vec{x}_n)_n$ be a seq. of sequences, $\vec{x}_n \in \mathbb{R}^{\mathbb{N}}$, we say $(\vec{x}_n)_n$ converges to $\vec{x} \in \mathbb{R}^{\mathbb{N}}$, $\boxed{\text{pointwise}}$ if $\forall i \in \mathbb{N}$, we have

$$\lim_{n\to\infty} x_{n,i} = x_i$$

$$\begin{array}{llllll}
\vec{x}_1 = & x_{11}, & x_{12}, & x_{13}, & x_{14}, & \cdots \\
\vec{x}_2 = & x_{21}, & x_{22}, & x_{23}, & x_{24}, & \cdots \\
\vec{x}_3 = & x_{31}, & x_{32}, & \cdots & & \\
\vdots & & & & x_{ni} & \\
 & & \downarrow & & & \\
\vec{x} = & x_1, & x_2, & x_3, & \cdots & x_i.
\end{array}$$

Def: Let $(\vec{x}_n)$ be a seq. of seq., $\vec{x}_n \in \mathbb{R}^{\mathbb{N}}$.
We say $\vec{x}_n \to \vec{x}$, $\boxed{\text{uniformly}}$, if

$$d_\infty(\vec{x}_n, \vec{x}) \to 0 \quad \text{as } n\to\infty.$$

i.e. $\forall \varepsilon > 0$, $\exists N > 0$, s.t. $\forall n > N$,

$$\sup\{|x_{ni} - x_i| : i \in \mathbb{N}\} < \varepsilon.$$

Ex: $x_{ni} = \dfrac{i}{n+i}$

$$\begin{array}{llll}
\frac{1}{2} & \frac{2}{3} & \frac{3}{4} & \cdots\cdots \xrightarrow{\text{limit}} 1. \\
\frac{1}{3} & \frac{2}{4} & \frac{3}{5} & \vdots \\
\frac{1}{4} & \frac{2}{5} & \frac{3}{6} & \vdots
\end{array}$$

$\frac{1}{5}$ $\frac{2}{6}$. $\frac{3}{7}$

$\downarrow$ lim=0. $\downarrow$

Pointwise, for fixed i, $\lim_{n\to\infty} x_{ni} = \lim_{n\to\infty} \frac{i}{n+i} = 0$

So, $\vec{x}_n$ converges to $\vec{0}$, pointwise.

$(0,0,0,\cdots)$

However. $d_\infty(\vec{x}_n, \vec{0})$

$= \sup.\{|x_{ni}| : i \in \mathbb{N}\}$.

$= \sup\{\frac{i}{n+i} : i \in \mathbb{N}\}$ · n is fixed in this set.

$= 1.$ · sup of ~~each~~ a given row

So, $\vec{x}_n$ do not converge to $\vec{0}$, uniformly.

- Sequence of functions, $Map(\mathbb{R}, \mathbb{R})$.
  - Given two functions $f, g : \mathbb{R} \to \mathbb{R}$., we define the "sup" distance.

$d_\infty(f,g) = \sup\{|f(x) - g(x)| : x \in \mathbb{R}\}$.

$f_n : \mathbb{R} \to \mathbb{R}$.

- Given a sequence of functions $f_n \in Map(\mathbb{R}, \mathbb{R})$.,

we say f_n converges to f pointwise, if $\forall x \in \mathbb{R}$,

$\lim_{n\to\infty} f_n(x) = f(x)$. $\left(\Leftrightarrow_{\text{i.e.}} \lim_{n\to\infty} |f_n(x) - f(x)| = 0.\right)$.

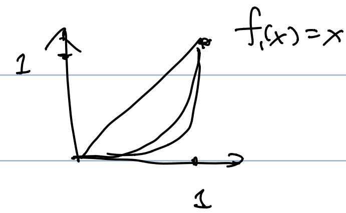
we say f_n converges to f uniformly, if

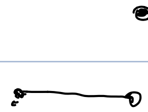
$$\lim_{n\to\infty} d_\infty(f_n, f) = 0.$$

Example of Pointwise convergence:

- $f_n : [0,1] \to \mathbb{R}$. $\quad f_n(x) = x^n$.

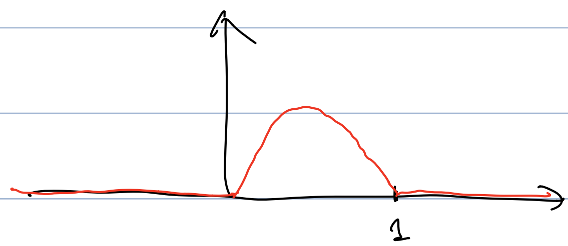
$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & x \in [0,1) \\ 1. & x = 1 \end{cases}$$





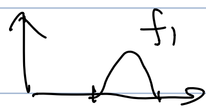
graph of the limit $f(x)$.

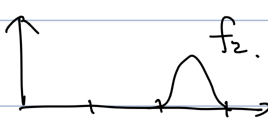
- (running bumps) : $\varphi(x) = \begin{cases} x(1-x) & x \in [0,1]. \\ 0 & \text{elsewhere} \end{cases}$



actually, any continuous function $\varphi(x)$, s.t. $\varphi(x) = 0 \quad \forall x \notin [0,1]$. is OK.

$$f_n(x) = \varphi(x-n).$$





$$\forall x, \quad \lim_{n\to\infty} f_n(x) = 0. = f(x)$$

$$\int_{\mathbb{R}} f_n(x)\, dx = C \qquad \forall n.$$

$$\neq \int_{\mathbb{R}} f(x)\, dx = 0.$$

• (Shrinking bump): with the same φ,

$$f_n(x) = n \cdot \varphi(nx).$$

then. $\forall x \in \mathbb{R}$, $\lim_{n \to \infty} f_n(x) = 0 =: f(x)$