Today: ① unif convergence

② unif convergence and continuity.

Def: A sequence of fcn $f_n : X \to \mathbb{R}$ (X: set), is said to converge uniformly to $f: X \to Y$, if $\forall \varepsilon > 0$, $\exists N > 0$, s.t. $\forall n > N$, $\forall x \in X$, we have

$$|f_n(x) - f(x)| < \varepsilon.$$

Compare with pointwise convergence $f_n(x) \to f(x)$, $\forall x$.

$\forall \varepsilon > 0$, $\forall x \in X$, $\exists \underline{N_{x,\varepsilon}} > 0$. s.t $\forall n > N_{x,\varepsilon}$, we have

$$|f_n(x) - f(x)| < \varepsilon.$$

Thm: ($\overset{\text{Unif}}{\text{Cauchy}} \Rightarrow \overset{\text{Unif}}{\text{Convergence}}$). Suppose $f_n : X \to \mathbb{R}$ satisfies that, $\forall \varepsilon > 0$, $\exists N > 0$, s.t. $\forall n, m > N$, $\forall x \in X$,

$$|f_n(x) - f_m(x)| < \varepsilon.$$

Then f_n converges uniformly.

Pf: • $\because \forall x \in X$, the seq of numbers $f_n(x)$ is a Cauchy seq, hence $\lim_{n\to\infty} f_n(x)$ exists. denote it as $f(x)$. We get $f: X \to Y$ as the pointwise limit.

• Let $\varepsilon > 0$ be given. Let N be given such that $\forall n, m > N$,

$$|f_n(x) - f_m(x)| < \varepsilon \quad \forall x \in X.$$

Then, taking $m \to \infty$, since $f_m(x) \to f(x)$, we have.

$$\underbrace{\lim_{m\to\infty} |f_n(x) - f_m(x)|} = \underbrace{|f_n(x) - f(x)|}.$$

Hence. $|f_n(x) - f(x)| \leqslant \varepsilon$. #

Rmk: Unif Cauchy $\Leftarrow$ Unif Convergence, by triangle inequality.

· Thm: Suppose $f_n \to f$ pointwise, then.

$$f_n \to f \text{ uniformly} \iff \lim_{n\to\infty} \Big(\overbrace{\sup_x |f_n(x) - f(x)|}^{d_\infty(f_n, f)} \Big) = 0.$$

· Thm: (Weierstrass M-test). Suppose $\underbrace{f(x) = \sum_{n=1}^{\infty} f_n(x)}$, $\forall x \in X$.

If $\exists M_n > 0$, s.t. $\sup_x |f_n(x)| \leq M_n$, and $\sum_n M_n < \infty$.

Then, the partial sum $F_N(x) = \sum_{n=1}^{N} f_n(x)$, converges to. $f(x)$ uniformly.

Pf: we only need to prove that $\{F_N\}$ forms a. uniform Cauchy sequeme.

$N < M$.

$$(*) \quad \sup_x |F_N(x) - F_M(x)| = \sup_x \Big|\sum_{n=N+1}^{M} f_n(x)\Big| \leq \sup_x \sum_{N+1}^{M} |f_n(x)|$$

$$= \sum_{N+1}^{M} \sup_n |f_n(x)| \leqslant \sum_{n=N+1}^{M} M_n.$$

∵ since $\sum M_n$ is convergent, hence $\forall \varepsilon > 0$, $\exists K > 0$.

s.t. $\forall N, M > K$. $N < M$. $\sum_{n=N+1}^{M} M_n < \varepsilon$., hence.

$$\sup_x |F_N(x) - F_M(x)| < \varepsilon \quad \forall N, M > K.$$

$\Rightarrow$ F_N is unif cauchy $\Rightarrow$ F_N converges uniformly.

#.

Unif Convergence and Continuity.

- Let X be a metric space, $E \subset X$.

Let $f_n : E \to \mathbb{R}$ be a seq of $\mathbb{R}$-valued fcn.

(7.11)

Thm: Suppose $f_n \to f$ uniformly on E in a metric ~~one~~ space X, Let x be a limit point of E., and suppose that.

$$\lim_{t \to x} f_n(t) = A_n.$$

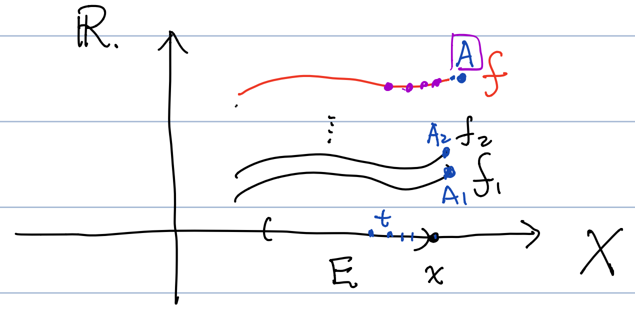
Then.

① the sequence A_n converges. Let $A = \lim_n A_n$.

②. $\lim_{t \to x} f(t) = A$.

In other words:
$$\lim_{t \to x} \underbrace{\lim_{n \to \infty} f_n(t)}_{f(t)} = \lim_{n \to \infty} \underbrace{\lim_{t \to x} f_n(t)}_{A_n}$$

Intuition:



- Fix $t_1, t_2, \cdots, t_n \to x$. Consider a double sequence

$$
\begin{array}{llllll}
f_1 & f_1(t_1), & f_1(t_2), & \cdots & & \longrightarrow A_1 \\
f_2 & f_2(t_1) & f_2(t_2), & \cdots & & \longrightarrow A_2 \\
f_3 & f_3(t_1) & f_3(t_2) & \cdots & & \longrightarrow A_3 \\
\vdots & & & & f_n(t_m) & \text{------} \; A_n \\
\text{unif convergence} \downarrow & \downarrow & \downarrow & & \Big| & \Big\| \; \textcircled{1} \\
f & f(t_1), & f(t_2), & \cdots & \underline{f(t_m)} & \cdots \overset{\textcircled{2}}{\Longrightarrow} \underline{A}
\end{array}
$$

Pf.: (1). Let $\varepsilon > 0$ be given. By unif convergence of f_n, $\exists N > 0$, s.t. $\forall\, n, m > N$, we have

$$(*) \qquad |f_n(t) - f_m(t)| < \varepsilon. \qquad \forall t \in X.$$

Let $t \to x$ in $(*)$, (we fix n, m in this limit), we have

$$\lim_{t \to x} |f_n(t) - f_m(t)| = |A_n - A_m|.$$

$$\Rightarrow \quad |A_n - A_m| \leq \varepsilon. \qquad , \qquad \forall\, n, m > N.$$

This shows $\{A_n\}$ is a Cauchy sequence, hence convergent. Let $A = \lim_{n} A_n$.

(2). $|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|$. $(\forall t \in E,\ n \in \mathbb{N})$.

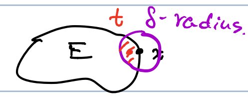
Then, we can choose n large enough, s.t.

$$|f(t) - f_n(t)| < \frac{\varepsilon}{3}. \qquad \forall t.$$

This is possible by unif convergence, ~~as~~ and also

$$|A_n - A| < \frac{\varepsilon}{3}.$$

Now, for this fix n, since $\lim_{t \to x} |f_n(t) - A_n| = 0$, there exists a $\delta > 0$. s.t. $\forall\, t \in E$, $0 < d(t,x) < \delta$, $t \neq x$



we have $|f_n(t) - A_n| < \frac{\varepsilon}{3}$.

Then, we have the following: given any $\varepsilon > 0$, we found a $\delta > 0$.

s.t. $\forall\, t \in E$, $t \neq x$, $0 < d(t,x) < \delta$,

$$|f(t) - A| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This shows $\lim_{t \to x} f(t) = A$. #.

<u>Thm</u>: Suppose $\{f_n\}$ is a seq of continuous fcn on E, and $f_n \to f$ uniformly, then f is continuous.

<u>Pf</u>: To show f is continuous, just need to show, $\forall\, x \in E' \cap E$. limit point of E, in E, we have

$$\lim_{t \to x} f(t) = f(x).$$

$\because f_n$ is cnt.

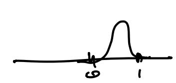
Then $f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{t \to x} f_n(t)$.

$$\overset{\text{by unif conv.}}{=} \lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{t \to x} f(t). \quad \#.$$

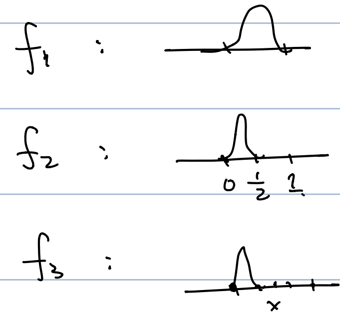
$f_n \to f$ ptwise.

<u>Q</u>: if $f_n : K \to \mathbb{R}$, continuous, is defined on a compact subset $K \subset X$. and $f_n(x) \to f(x)$, $\forall\, x \in K$.

Do we have $f_n \to f$ uniformly (indep of $x \in K$).

<u>A</u>: . consider $\varphi(x) : \mathbb{R} \to \mathbb{R}$. a bump function supported on $[0,1]$, i.e. $\varphi(x) \geq 0$,  , φ continuous

· Let $f_n(x) = \varphi(nx)$.



· $\forall x \in [0,1]$, $f_n(x) \to 0$.

So. f_n restricted on $[0,1]$. converge ptwise to the 0- function.

But. $\sup\limits_{x \in [0,1]} |f_n(x) - 0| = C = \sup\limits_{x} |\varphi(x) - 0|$

hence $\{f_n\}$ do not converge to 0 uniformly.

Thm: Suppose K compact. and.

(a). $\{f_n\}$. is a seq of cont. fcn on K.

(b). $\{f_n\}$ converges pointwise to a continuous function $f(x)$ on K.

(c). $f_n(x) \geq f_{n+1}(x)$. $\forall x \in K$, $\forall n = 1, 2, \cdots$.

Then, $f_n \to f$ uniformly on K.

Pf: Let $g_n(x) = f_n(x) - f(x)$. Then. $g_n(x) \to 0$, ptwise,

$g_n(x)$ continuous, $g_n(x) \geq g_{n+1}(x)$. We only need to prove.

$g_n \to 0$ uniformly on K.

Let $\varepsilon > 0$ be given. Let $K_n = \{ x \in K \mid g_n(x) \geq \varepsilon \}$

(these are the bad set ...). $g_n(x) < \varepsilon \ \forall x \in K \iff K_n = \phi$.

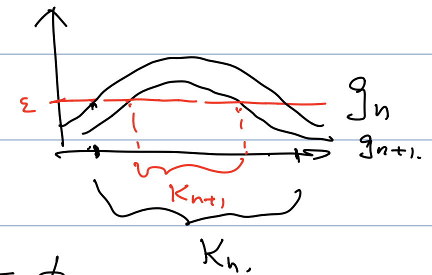
· Since g_n is continuous, $K_n = g_n^{-1}(\underbrace{[\varepsilon, +\infty)}_{\text{closed}})$ is closed.

· Since $K_n \subset K$ is a closed set in a compact set, $\therefore K_n$ is also compact.

· since $g_n(x) > g_{n+1}(x)$, $K_n \supset K_{n+1} \cdots$.

For any $x \in K$, since $g_n(x) \to 0$,

hence $x \notin \bigcap_{n=1}^{\infty} K_n \Rightarrow \bigcap_{n}^{\infty} K_n = \phi$.



$\Rightarrow \exists N$, s.t. $\bigcap_{n=1}^{N} K_n = \phi \Rightarrow K_N = \phi$.

Thm 2.36 $\Rightarrow$ [Otherwise, if $\bigcap_{n=1}^{N} K_n \neq \phi \ \forall N$, we can construct a seq. $x_N \in \bigcap_{n=1}^{N} K_n$, get convergent subseq.

$\Rightarrow x$ lim of subseq. s.t. $x \in \bigcap_{n=1}^{\infty} K_n$,

a contradiction.]

$\Rightarrow \forall n > N$, $g_n(x) < \varepsilon$.

This shows $g_n \to 0$ uniformly. #