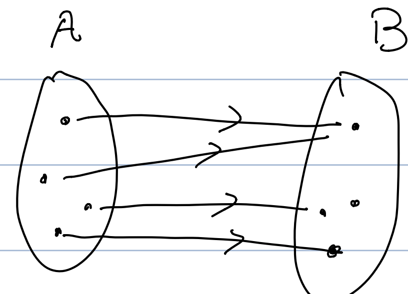
- Continuous functions. (from metric space to metric space)

Follow Rudin Ch 4. Ch 7
↗ uniform convergence.

<u>Recall</u> : A function $f$ from set $A$ to set $B$.

is an assignment. for each element $\alpha \in A$, an element $f(\alpha)$ in $B$.




- $f$ is <u>injective</u> (or one-to-one onto its image), if $\forall\, x, y \in A$. $x \neq y$. then $f(x) \neq f(y)$

$f: A \hookrightarrow B$

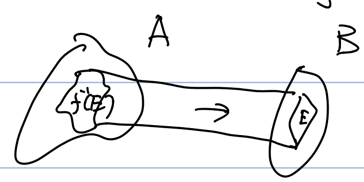
- $f$ is <u>surjective</u>, if $\forall\, \beta \in B$, there exists at least one element $\alpha \in A$, s.t. $f(\alpha) = \beta$.

$f: A \twoheadrightarrow B$

- $f$ is <u>bijective</u>, if $f$ is both injective and surjective.
- Given a subset $E \subset B$, $f^{-1}(E)$ is a subset of $A$.

$$f^{-1}(E) = \{ \alpha \in A \mid f(\alpha) \in E \}.$$

↗ "pre-image" of E"

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- Let $(X, d_X)$, $(Y, d_Y)$ be metric spaces.

Let $E \subset X$. $f: E \to Y$.

- Recall $E'$ is the set of limit points of $E$.

$$E' = \{ x \in X \mid \forall \varepsilon > 0,\ \exists\, y \in E, \text{ s.t. } y \neq x,\ d_X(y, x) < \varepsilon \}.$$

$$= \{ x \in X \mid \forall\, \varepsilon > 0,\ \mathring{B}^X_\varepsilon(x) \cap E \neq \phi \}$$

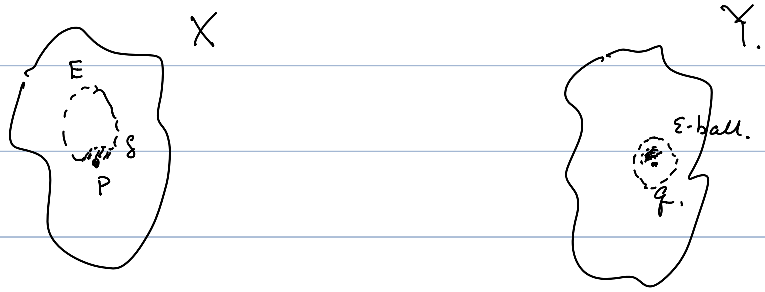
↑ punctured open ball centered at $x$.

- <u>Def</u> (limit of function). Suppose $p \in E'$. We write $f(x) \to q$ ($q \in Y$)

as $x \to p$, or $\lim_{x \to p} f(x) = q$.

if $\forall \varepsilon > 0$, $\exists\, \delta > 0$, such that

- $\forall x \in E$, $0 < d_X(x, p) < \delta \Rightarrow d_Y(f(x), q) < \varepsilon$.




Thm: with the same notation as above. The following are equivalent

(1) $\lim_{x \to p} f(x) = q$

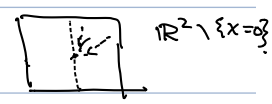
(2) for any sequence of points in E convergent to p.

$(p_n)$, $\lim_{n \to \infty} p_n = p$, we have $\lim_{n \to \infty} f(p_n) = q$.

(Read Rudin thm 4.2)

Corollary: if $f$ has a limit at point $p$, then it is unique.

Ex: $f: \mathbb{R}^2_{x \neq 0} \to \mathbb{R}$. $f(x,y) = \frac{y}{x}$. at $p = (0,0)$, $f$ does not have a limit




Rmk: • $E'$ is not necessarily contained in $E$.

e.g. $E = (0,1)$, $E' = [0,1]$.

$E = \{\frac{1}{n}, n \in \mathbb{N}\}$, $E' = \{0\}$.

(4.4).

Thm: Suppose we have $f, g : E \to \mathbb{R}$. suppose

$p \in E'$, and $\lim_{x \to p} f(x) = A$, $\lim_{x \to p} g(x) = B$, then.

(1) $\lim_{x \to p} f(x) + g(x) = A + B$

(2) $\lim_{x \to p} f(x)\, g(x) = A \cdot B$

(3) $\lim f(x) / g(x) = A / B$ if $B \neq 0$ and $g(x) \neq 0$ $\forall x \in E$.

$x \to p$

(4). $\forall c \in \mathbb{R}, \quad \lim_{x\to p} c\cdot f(x) = c\cdot A.$

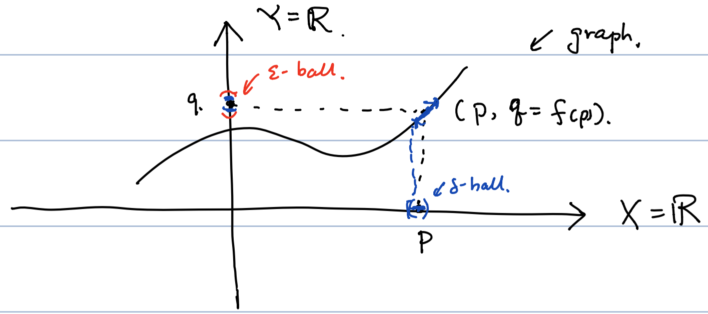
Pf: Use the alternative definition of $\lim_{x\to p} f(x)$ using seq., we reduces these claims to corresponding claims for convergents sequence. #.

## Continuity of functions.

Def (continuity at a point). Let $(X, d_X)$, $(Y, d_Y)$ be metric spaces. $E \subset X$. $f: E \to Y$. Let $p \in E$, and $q = f(p)$. We say $f$ is continuous at $p$, if $\forall \varepsilon > 0$, $\exists \delta > 0$, such that

$$\forall x \in E, \text{ with } d_X(x,p) < \delta, \Rightarrow d_Y(f(x), q) < \varepsilon.$$

$\Leftarrow$ in other words, $\underline{f(B_\delta(p))} \subset B_\varepsilon(q)$.




Thm: If $p \in E$ is also a limit point of $E$, then

$$f \text{ is continuous at } p \Leftrightarrow \lim_{x\to p} f(x) = f(p).$$

Pf: Exercise.

Rmk: (1) $p \in E$ can either be a limit or an isolated point. The condition of continuity is automatically satisfied for isolated pt. Hence continuity condition is only nontrivial for limit point of $E$, inside $E$.

Def: We say $f$ is continuous on $E$, if $f$ is continuous at every point in $E$.

(3rd definition of continuity).

Thm: $(X, d_X)$, $(Y, d_Y)$, $f: X \to Y$ as above.
Then $f$ is continuous if and only if, for every openset $V \subset Y$, $f^{-1}(V)$ is open in $X$.

($f$ is cont. if preimage of open is open).

Pf: Suppose $f$ is continuous. We need to show that, $\forall V \subset Y$ open, $f^{-1}(V)$ is open. We need to show, $\forall p \in f^{-1}(V)$, $\exists \delta > 0$, s.t. $B_\delta(p) \subset f^{-1}(V)$. Since $f(p) \in V$, and $V$ is open, we have $\varepsilon > 0$, s.t. $B_\varepsilon(f(p)) \subset V$. By definition of continuity of $f$ at $p$, we have a $\delta > 0$, s.t. $f(B_\delta(p)) \subset B_\varepsilon(f(p))$. $\Leftrightarrow$ $B_\delta(p) \subset f^{-1}(B_\varepsilon(f(p)))$

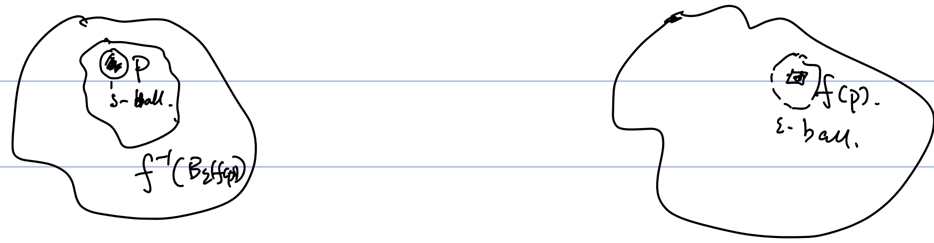
$\subset f^{-1}(V)$. ✓.

Suppose $\forall V \subset Y$ open, $f^{-1}(V)$ is open. We need to check that $\forall p \in X$, $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t.

$$f(B_\delta(p)) \subset B_\varepsilon(f(p)) \Leftrightarrow B_\delta(p) \subset f^{-1}(B_\varepsilon(f(p))).$$

$\because$ $B_\varepsilon(f(p))$ is open, we can take $V = B_\varepsilon(f(p))$, and get $f^{-1}(B_\varepsilon(f(p)))$ is open in $X$. Hence there exists $\delta > 0$, s.t. $B_\delta(p) \subset f^{-1}(B_\varepsilon(f(p)))$. #.

X Y




**Lemma**: if $f: A \to B$ is a function, and $E \subset A$, $F \subset B$. Then. $f(E) \subset F \iff E \subset f^{-1}(F)$.

Pf: $f(E) \subset F \iff \forall x \in E,\ f(x) \in F$

$\iff \forall x \in E,\ x \in f^{-1}(F)$.

$\iff E \subset f^{-1}(F)$.

Thm: Let $X, Y, Z$ be metric spaces, and.

$f: X \to Y$, $g: Y \to Z$ continuous function.

We define $h: X \to Z$, by $h(x) = g(f(x))$.

Then $h$ is also continuous. ( $h = g \circ f$ notation "composition" )

Pf: $\forall V \subset Z$ open, we have $g^{-1}(V)$ is open in $Y$.

and $f^{-1}(g^{-1}(V))$ is open in $X$.

But, $h^{-1}(V) = f^{-1}(g^{-1}(V))$, hence $h^{-1}(V)$ is open in $X$.

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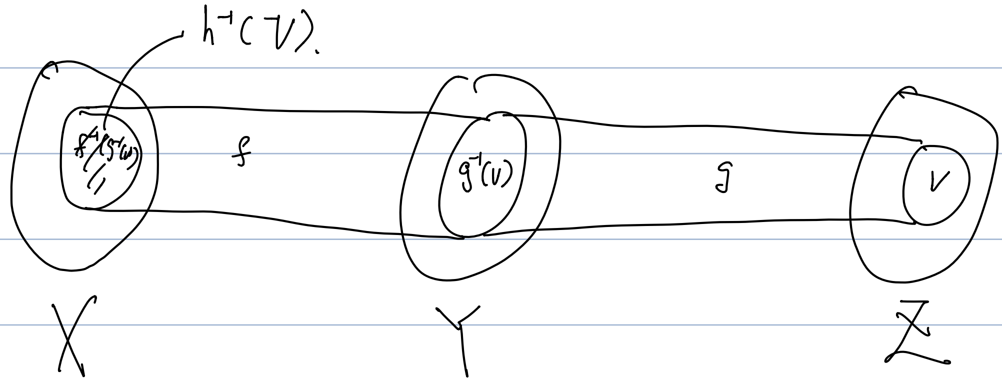

**Thm:** If $f, g : X \to \mathbb{R}$ continuous.

then $f+g$, $f-g$, $f \cdot g$ are continuous functions.

and if $g(x) \neq 0$ for any $x \in X$, then $f/g$ is continuous.

**Pf:** We prove here $f+g$ is a continuous function.

For any $p \in X'$, a limit point $X$, we need to check

that $f(p)+g(p) = \lim_{x \to p} (f(x)+g(x))$, this follows from

$\lim_{x\to p} f(x) = f(p)$ and $\lim_{x \to p} g(x) = g(p)$ and Thm 4.4 (Rudin).

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**Ex:** (1) $X : \mathbb{R} \to \mathbb{R}$ is a continuous function.

send a point to itself.

(2). $X^2 : \mathbb{R} \to \mathbb{R}$ is continuous. $X^2 = X \cdot X$.

$\rightsquigarrow$ $X^n : \mathbb{R} \to \mathbb{R}$ is continuous.

$\rightsquigarrow$ $P(X) := a_n X^n + \cdots + a_0 : \mathbb{R} \to \mathbb{R}$ is continuous.

**Thm:** let $f : X \to \mathbb{R}^n$, with components of $f$.

$$f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$$

Then. $f$ is continuous, if and only if each $f_i$ is continuous.

**Pf:** exercise.

Hint:

$f$ continuous $\Rightarrow$ $f_i$ is continuous

$\because$ $f_i = \pi_i \circ f$

$\pi_i : \mathbb{R}^n \to \mathbb{R}$ $\quad i \in \{1, \ldots, n\}$.

$(x_1, \ldots, x_n) \mapsto x_i$ continuous.