Today:
- brief recap
- Q & A
- sample problem.

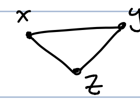
- Series & Convergence (10%).

  ratio test, root test, integral test, alternating series test

- Metric Space & Topology (Ross §13, ⋯ ; Rudin Ch 2)

metric space (X, d): data: X set, $d: X \times X \to \mathbb{R}$

condition: ① ② ③ $d(x,y) \le d(x,z) + d(y,z)$.



induced metric: If (X, d_X) met. sp., $S \subset X$ subset., we can endow S with a distance function. $d_S: S \times S \to \mathbb{R}$

$$d_S(x,y) = d_X(x,y) \quad \forall\, x, y \in S.$$

Topology associated to a metric space: Let (X,d) be a met. sp.

$U \subset X$ is open iff $\forall p \in U$, $\exists\, \delta > 0$, s.t. $B_\delta(p) \subset U$.

$$B_\delta(p) = \{x \in X \mid d(p,x) < \delta\}.$$

The collection of open sets satisfies:

① X, ϕ are open subset

② arbitrary union of open is open

③ finite intersection of open is open.

Hence, we get a topological space X.

- Closed set. $E \subset X$ is closed iff $\underset{\overset{\shortparallel}{X \setminus E}}{E^c}$ is open.

① X, ϕ are closed

② arb intersection of closed is closed

③ finite union of closed is closed.

- Compact Set: (X,d) metric space.

Def: $K \subset X$ is compact, if any open cover of K admits a finite subcover.

(i.e. For any collection of open sets $\{U_i\}_{i\in I}$, s.t. $U_i \subset X$ open. $K \subset \bigcup_{i\in I} U_i$, there exist a finite subset $A \subset I$, s.t. $K \subset \bigcup_{i\in A} U_i$.)

Def (seq. compactness): $K \subset X$ is seq compact, if for any seq (x_n) in K, there is a convergent subsequence $(x_{n_k})_k$, s.t. $x_0 = \lim_k x_{n_k}$ exists and is in K.

Thm: K is compact $\Leftrightarrow$ K is seq compact.

If $X = \mathbb{R}^n$, then. $K \subset X$ compact $\Leftrightarrow$ K is closed and bounded.

Ex: $X = (0,1)$. Then if set $K = (0,1)$.

K is bounded $\Leftrightarrow$ $\sup_{x,y\in K} d(x,y) < \infty$. ✓

K is closed in X.

$\Big\}$ K is not compact.

Rmk: ① open and closed are relative notion, one need to specify the ambiant space.

Compactness is "absolute" notion, independent of the ambient space.

② For general metric space. (X, d), ~~but~~ we still have

K compact $\Rightarrow$ K is bounded and closed.

$$\left.\begin{array}{l} K \subset X \text{ compact} \\ E \subset X \text{ closed} \end{array}\right\} \Rightarrow K \cap E \text{ is compact.}$$

Connectedness:

- Let X be a top. space. X is connected iff X cannot be written as a disjoint union of two non-empty open subsets in X.

(different but equivalent to Rudin's def'n.).

- $E \subset \mathbb{R}$ is connected $\Leftrightarrow$ $\forall x < y$ in E, $[x, y] \subset E$.

Hence connected subset are intervals.

(a, b)
$(a, b]$
$\vdots$
(a, ∞).

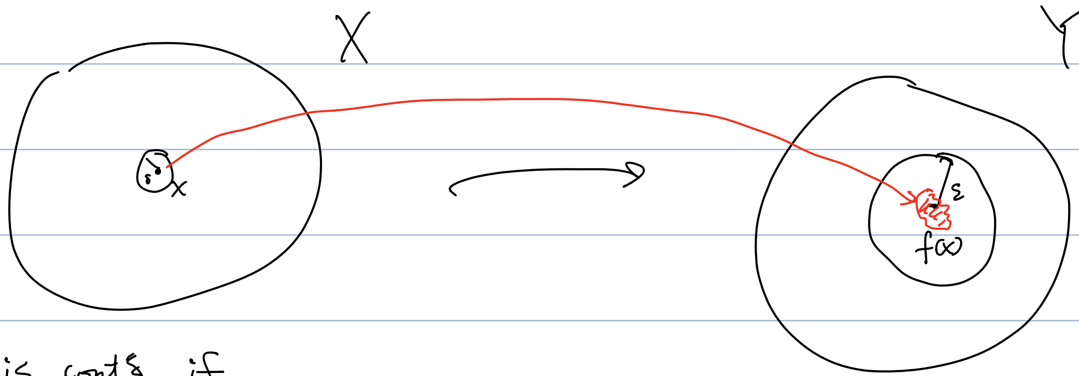
- Continuous Maps between metric space (or topological spaces).

Let (X, d_X), (Y, d_Y) be met. space.

$f : X \to Y$ is a map.

Def1 : f is continuous, if $\forall x \in X$, $\forall \varepsilon > 0$, $\exists \delta > 0$.

s.t. $f(B_\delta(x)) \subset B_\varepsilon(f(x))$



f is cont. if

Def2 : $\forall x \in X$, for all sequence $x_n \to x$, we have,

$$\lim_{n\to\infty} f(x_n) = f(x).$$

Equivalently, only need to check for $x \in X$ limit point of X,

for seq $x_n \to x$ that $x_n \neq x$, we have $\lim_{n\to\infty} f(x_n) = f(x)$.

f is cont, if

Def3 : $\forall V \subset Y$ open subset, $f^{-1}(V)$ is open.

Def 3' : f is cont if $\forall E \subset Y$ closed, $f^{-1}(E)$ is closed.

Prop : Given $f: X \to Y$ continuous:

(1) if $K \subset X$ is compact, then $f(K)$ is compact

(2) if $K \subset X$ is connected, then $f(K)$ is connected.

Rmk : (1) if $U \subset X$ is open. $f(U)$ may not be open.

$f(x) = x^2$, $U = (-1, 1)$ $f(U) = [0, 1)$

(2). if $K \subset Y$ is compact, $f^{-1}(K)$ may not be compact.

Ex: $f: (0,1) \to \mathbb{R}$ inclusion.

$K = [0,1]$ in $\mathbb{R}$, $f^{-1}(K) = (0,1)$ not compact.

<u>Uniform Continous function</u>: Let X, Y be metric spaces.

$f: X \to Y$ is unif cont. if $\forall \varepsilon > 0, \exists \delta > 0$. s.t.

$\forall x, y \in X$, with $d_X(x,y) < \delta$, their images

$d_Y(f(x), f(y)) < \varepsilon$.

"Free Upgrade": if $f: X \to Y$ cts, X cpt

then f is unif. cts.

<u>Ex of non-unif cont.</u>: ∘ $f(x) = \sin(\frac{1}{x})$ $x \in (0, \infty)$

∘ $f(x) = x^2$ $x \in \mathbb{R}$

<u>Ch 4 of Rudin ↑↑</u>.

metric sp.

$f_n : X \to \mathbb{R}$

$f : X \to \mathbb{R}$

• Sequence of functions and convergences.

Different versions of convergences

$f_n \to f$ pointwise $\iff \forall x \in X$, $\lim_{n \to \infty} |f_n(x) - f(x)| = 0$

$f_n \to f$ uniformly $\iff \lim_{n \to \infty} \sup_{x \in X} |f_n(x) - f(x)| = 0$

uniform convergence is a stronger notion that ptwise convergence.

<u>Ex</u>: say $\varphi(x)$ is a bump function 

• $f_n(x) = \varphi(x - n)$. $f_n \to 0$ ptwise not uniformly

- $f_n(x) = \varphi(n \cdot x)$. $f_n \to 0$ ptwise, not uniformly.

- $f_n(x) = \frac{x}{n}$, $f_n \to 0$ ptwise, not unif.

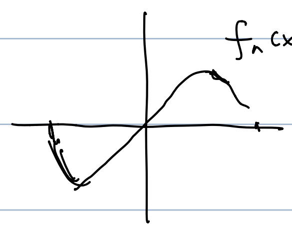
Prop: If $f_n : K \to \mathbb{R}$ is a seq of $\overset{\text{conts.}}{\text{fcn}}$ on a compact set K, such that

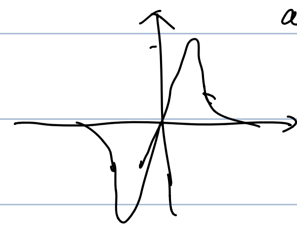
- $f_n \to f$ ptwise, and f is continuous.
- $f_{n+1}(x) \le f_n(x)$ $\forall x \in K$.

Then, $f_n \to f$ uniformly.

Ex: $f_n(x) = \dfrac{x}{1 + n \cdot x^2}$ over $[-1, 1]$.

Pf #1

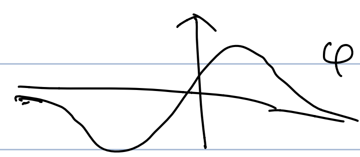




$$f_n(x) = \frac{1}{\sqrt{n}} \frac{(\sqrt{n}\, x)}{1 + (\sqrt{n}\, x)^2} = \underline{\frac{1}{\sqrt{n}} \varphi(\sqrt{n}\, x)}$$

$$\varphi(u) = \frac{u}{1 + u^2}.$$

$\rightsquigarrow$ $\underline{f_n \to 0 \text{ uniformly}}$,



since,

$$\sup_x |f_n(x) - 0| \le \frac{1}{\sqrt{n}} \underbrace{\sup_{x \in \mathbb{R}} |\varphi(x)|}_{\text{a const}}$$

Pf #2: since $|f_n(x)| = \dfrac{|x|}{1 + n x^2}$ satisfies the prop's condition, ~~...~~ hence $f_n \to 0$ uniformly.

Rudin's def'n of "limit point" of a subset $E \subset X$.

- $x \in X$ is a <u>limit point of E</u>, if $\forall \delta > 0$, $B^{\times}_{\delta}(x) \cap E \neq \phi$.

$$B^{\times}_{\delta}(x) = B_{\delta}(x) \setminus \{x\} = \{ y \in X \mid 0 < d(y,x) < \delta \}.$$

- x is a limit pt of E $\iff$ $\exists\, x_n \to x$, s.t. $x_n \in E$, $\underline{x_n \neq x}$.

Sample Problems:

(1). Let X be a complete metric space. (e.g. $X = \mathbb{R}^n$),

For each $n \in \mathbb{N}$, let $O_n \subset X$ be an open dense subset. Then $\bigcap_{n=1}^{\infty} O_n$ is dense in X.

> $E \subset X$ is dense iff $\overline{E} = X$.
>
> i.e. $\forall x \in X$, $\forall \delta > 0$. $B_{\delta}(x) \cap E \neq \phi$.
>
> i.e. $\forall x \in X$, $\forall \delta > 0$. $\exists x'$, s.t. $d(x, x') < \delta$. and $x' \in E$.

obs: if we drop the open requirement of O_n, this fails.. e.g. $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$ but have empty intersections.

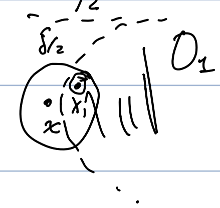
- pick any $x \in X$, we need to show that, $\forall \delta > 0$, $\exists z \in X$, s.t. $d(z, x) < \delta$. and $z \in \bigcap_{n=1}^{\infty} O_n$.

- We are going to construct a seq $x_1, x_2, x_3, \cdots$.

s.t. $\quad x_n \in O_1 \cap O_2 \cap \cdots \cap O_n.$

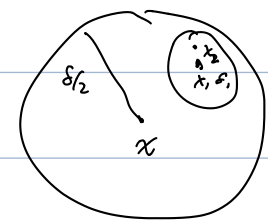
and $\quad d(x_n, x) < \delta/2.$

For x_1, by denseness of O_1, $\exists\, x_1 \in B_{\delta/2}(x) \cap O_1$.



Choose $\delta_1 > 0$ small enough, such that

- $\overline{B_{\delta_1}(x_1)} \subset O_1$
- $\overline{B_{\delta_1}(x_1)} \subset B_{\frac{\delta}{2}}(x)$
- $\delta_1 \leq \delta/2.$



For x_2, we choose $x_2 \in B_{\delta_1}(x_1) \cap O_2 \neq \phi$

Then $\quad x_2 \in B_{\delta_1}(x_1) \subset B_{\delta/2}(x)$.

find $\delta_2 > 0$ small enough, s.t.

- $\overline{B_{\delta_2}(x_2)} \subset O_2$
- $\overline{B_{\delta_2}(x_2)} \subset B_{\delta_1}(x_1)$
- $\delta_2 \leq \delta_1/2$

Continuing this way, we get a sequence of points (x_n), and radii δ_n, s.t.

$$\left.\begin{array}{l} \cdot\ \delta_n \leq \delta_{n-1}/2 \\ \cdot\ \overline{B_{\delta_n}(x_n)} \subset B_{\delta_{n-1}}(x_{n-1}) \\ \cdot\ \overline{B_{\delta_n}(x_n)} \subset O_n. \end{array}\right\} \Rightarrow x_n \in O_1 \cap \cdots \cap O_n.$$

$(\because x_n \in B_{\delta_m}(x_m))$.

- since $\forall\, n > m$, $\quad d(x_n, x_m) \leq \delta_m$,

and $\delta_m \to 0$ as $m \to \infty$, $\quad (x_n)$ is a Cauchy seq.

Since X is complete, (x_n) has a limit, call it z. $\forall m \in \mathbb{N}$,

$$d(z, x_m) = \lim_{\substack{n \geq m \\ n \to \infty}} d(x_n, x_m) \leq \delta_m.$$

$$\Rightarrow \quad z \in \overline{B_{\delta_m}(x_m)} \subset O_1 \cap \cdots \cap O_m.$$

hence $\forall m$, $z \in O_1 \cap \cdots \cap O_m$.

Hence $z \in \bigcap_{n=1}^{\infty} O_n$

and $d(z, x) \leq \frac{\delta}{2} < \delta$. #. Baire's Theorem.