0. Review

- Metric Space (X, d)
- Topology on (X, d): $U \subset X$ is <u>open</u> if $\forall p \in U$, $\exists \delta > 0$, s.t. $B_\delta(p) \subset U$.
  - $V \subset X$ is <u>closed</u> if V^c is open.

- <u>Compact subset</u>: <u>Def</u>: $K \subset X$ is compact. if for any open cover of K, $\exists$ a finite subcover of K.

<u>Properties</u>:
- K compact $\Rightarrow$ K is bounded.
- K compact $\Rightarrow$ K is closed.
- $\left.\begin{array}{l} E \subset X \text{ closed}, \ K \text{ cpt}, \\ E \subset K \end{array}\right\} \Rightarrow E$ is also compact.

(see <u>Rudin 2.41</u>).

<u>Thm 1</u> $K \subset X$ is compact $\Leftrightarrow$ $\forall$ seq in K, $\exists$ convergent subseq in K (which requires the limit of the subseq in K).

<u>Thm 2</u> (Heine - Borel)

For $\mathbb{R}^n$, $K \subset \mathbb{R}^n$ is compact $\Leftrightarrow$ K is closed and bounded.

<u>Rmk</u>: open and closed, are "relative properties", i.e. we should say "E is an open subset <u>in space X</u>", we should specify the ambiant space.

<u>Recall</u>: (X, d) metric space, $S \subset X$.

then $(S, d_S : S \times S \to \mathbb{R})$ is a metric space.

"$d|_{S\times S}$" with the "induced metric".

- Open set in S may not be open set in X.

Ex: · $X = \mathbb{R}$, with $d(x,y) = |x-y|$.

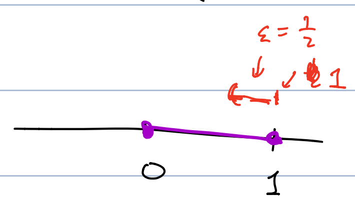
· $S = [0,1] \subset X$. with induced metric. d_S.

- What are open sets in S?

open ball $B^{(S)}_{\varepsilon}(p) = \{ q \in S \mid d_S(q,p) < \varepsilon \}$.

e.x.: $B^{(S)}_{\frac{1}{2}}(1) = \{ q \in [0,1] \mid |q-1| < \tfrac{1}{2} \}$

$= (\tfrac{1}{2}, 1]$.



$(\tfrac{1}{2}, 1]$ is open in S.

$(\tfrac{1}{2}, 1]$ is not open in X.

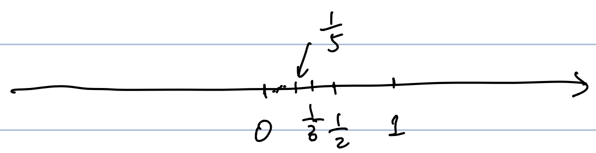
Ex: · $X = \mathbb{R}$, $S = \{ \frac{1}{n} : n \in \mathbb{N} \} \cup \{0\}$.

what are open sets in S?

Q1: is the set $\{\frac{1}{5}\}$ open? Yes.

is the set $\{ \frac{1}{n} \mid n \in \mathbb{N} \}$ open?

is the set $\{0\}$ open?



- $B^{(S)}_{\frac{1}{100}}(\tfrac{1}{5}) = \{ q \in S \mid |q - \tfrac{1}{5}| < \tfrac{1}{100} \} = \{\tfrac{1}{5}\}$. is open.

(in S)

· $\{\frac{1}{n}\}$ is open, similarly $\Rightarrow \bigcup_{n \in \mathbb{N}} \{\frac{1}{n}\}$ is open.

- $B_r^{(S)}(p) = \{ x \in S \mid d(x,p) < r \}$ open ball, radius r, center $= p$, in $\underline{\underline{S}}$.

- $B_\varepsilon^{(S)}(0) = \{ x \in S \mid \overset{|x|}{\overset{\shortparallel}{d(x,0)}} < \varepsilon \}$

contains $\frac{1}{n}$, for $n > \lfloor \frac{1}{\varepsilon} \rfloor$. Hence is not contained in $\{0\}$. Hence $\{0\}$ is not open.

- <u>Induced topology</u>: Say X is topological space, $S \subset X$. We can equip S with the induced topology, $E \subset S$, E is open in S iff there exist an open subset $\tilde{E} \subset X$, s.t. $E = S \cap \tilde{E}$.

Given (X, d), $S \subset X$, we get 2 ways of getting topologies on S.

$$
\begin{tikzcd}
 & (X, d) \text{ metric space} \arrow[dl, "\substack{\text{metric}\\ \text{inducing}\\ \text{topology}}"'] \arrow[dr, "\text{induced metric}"] & \\
X \text{ topological space} \arrow[dr, "\substack{\text{induced}\\ \text{topology}}"'] & & (S, d_S) \arrow[dl, "\substack{\text{metric inducing}\\ \text{topology}}"] \\
 & S &
\end{tikzcd}
$$

<u>They define the same topologies on S.</u>

- Back to example $S = \{ \frac{1}{n} \; n \in \mathbb{N} \} \cup \{0\}$.

$$\{\tfrac{1}{5}\} = S \cap \underbrace{(\tfrac{1}{5} - \varepsilon, \tfrac{1}{5} + \varepsilon)}_{\text{open in } X}$$

$\Rightarrow \{\frac{1}{5}\}$ is open in S.

- Inclusion map : $\iota : S \to X$.
  - preserves distance. $\Rightarrow$ ι is continuous.

<u>Rmk 2</u> : Compactness is an intrinsic notion.

* $\left\{\begin{array}{l}\text{If } K \subset X \text{ is a compact subset, then } \forall S \subset X, \\ \text{s.t. } K \subseteq S, \quad K \text{ is also a compact subset in } S.\end{array}\right\}$

Hence, we can simply say "K is compact", without say what is the ambient space.

Suppose $\iota : X \to Y$ inclusion map., with compatible topology, i.e. $\forall\ \mathcal{U} \subset X$ open, $\exists\ \tilde{\mathcal{U}} \subset Y$ open. s.t. $\mathcal{U} = X \cap \tilde{\mathcal{U}}$.

Then, claim $K \subset X$ compact $\Rightarrow$ $K \subset Y$ compact.

say $K \subset \bigcup_{\alpha \in A} V_\alpha$ $\quad V_\alpha \subset Y$ open.

then $K \subset \bigcup_{\alpha \in A} (V_\alpha \cap X)$ $\quad (V_\alpha \cap X)$ is open in X.

Hence we have a finite subcover, $I \subset A$ finite

$$K \subset \bigcup_{\alpha \in I} (V_\alpha \cap X) \subset \bigcup_{\alpha \in I} V_\alpha.$$

<u>Continuous Maps</u> : (X, d_X), (Y, d_Y) metric space.

$f : X \to Y$. map. (3 ways of definition of continuity.)

<u>Def 1</u> : f cont $\Leftrightarrow$ $\forall p \in X$, $\forall \varepsilon > 0$, $\exists \delta > 0$,

s.t. $\forall x \in X$, with $d_X(x, p) < \delta$, we have

$$d_Y(f(p), f(x)) < \varepsilon.$$

i.e., $f(B_\delta^{(X)}(p)) \subset B_\varepsilon^{(Y)}(f(p))$.

<u>Def 2</u>: f cont. $\Leftrightarrow$ $\forall\, V \subset Y$ open, $f^{-1}(V)$ is open in X.

<u>Def 3</u>: f cont $\Leftrightarrow$ $\forall$ convergent seq $x_n \to x$ in X

we have $f(x_n) \to f(x)$ in Y.

<u>Today</u>: <u>Thm</u>: $f: X \to Y$ cont.

$E \subset X$ compact

then $f(E) \subset Y$ is compact.

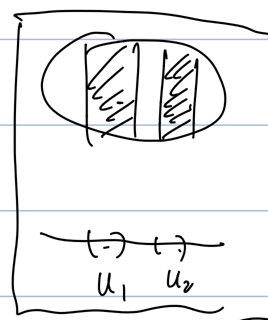
<u>Pf</u>: • To show $f(E)$ is compact, we need to show

for any open cover $f(E) \subset \bigcup_{\alpha\in A} U_\alpha$, $U_\alpha \subset Y$ open,

there exist $I \subset A$ finite. s.t.

$$f(E) \subset \bigcup_{\alpha\in I} U_\alpha.$$

• $f(E) \subset \bigcup_{\alpha\in A} U_\alpha \Leftrightarrow E \subset f^{-1}\left(\bigcup_{\alpha\in A} U_\alpha\right) \overset{\checkmark}{=} \bigcup_{\alpha\in A} f^{-1}(U_\alpha)$.



$\because f$ cont. $\therefore f^{-1}(U_\alpha)$ is open, $\forall \alpha$.

Hence $\{f^{-1}(U_\alpha)\}_{\alpha\in A}$ is an open cover of compact set E, hence admits a finite subcover.

$$E \subset \bigcup_{\alpha\in I} f^{-1}(U_\alpha) = f^{-1}\left(\bigcup_{\alpha\in I} U_\alpha\right)$$

$$\Rightarrow \quad f(E) \subset \bigcup_{\alpha\in I} U_\alpha. \quad \checkmark$$

#.

pf #2 : using compactness $\Leftrightarrow$ sequential compactness.

We need to show $f(E)$ is seq. cpt., i.e. $\forall (y_n)$ in $f(E)$, we need to show there exist subseq $(y_{n_k})_k$, s.t. $\lim_{k\to\infty} y_{n_k} = y \in f(E)$.

For each y_n in the seq, $\because y_n \in f(E)$ $\therefore \exists x_n \in E$, s.t. $f(x_n) = y_n$. Choose such a x_n for each y_n, $\forall n \in \mathbb{N}$. Then (x_n) is a seq in E. $\because$ E is seq compact $\therefore$ $\exists$ subseq $(x_{n_k})_k \to x \in E$. Apply f to this convergent subseq $(x_{n_k})_k$, we get by continuity,

$$f(x_{n_k}) \longrightarrow f(x) \quad \text{as } k\to\infty.$$

i.e., $$y_{n_k} \longrightarrow \overset{\shortparallel}{y} \quad \text{as } k\to\infty.$$ #.

Cor : If $f: X \to \mathbb{R}$ cont., if $E \subset X$ cpt. then, $\exists p, q \in E$, s.t. $f(p) = \sup(f(E))$, and $f(q) = \inf(f(E))$.

Pf : $f(E)$ is compact in $\mathbb{R}$ $\Rightarrow$ $f(E)$ closed and bounded in $\mathbb{R}$.

Let $A = \sup(f(E))$, by boundness of $f(E)$, $A \in \mathbb{R}$. By closedness of $f(E)$, $A \in f(E)$. Hence $\exists p \in E$, s.t. $f(p) = A$. The inf statement is similar. #.

Rmk: for K cpt in $\mathbb{R}$, we can ~~say~~ use $\max(K)$, and $\min(K)$.

Ex: $f : \overset{X}{\overset{\parallel}{(0,\infty)}} \to \mathbb{R}$. $\quad f(x) = \frac{1}{x}$.

- f is continuous since $1, x$ are cont., x is nonzero function on $(0,\infty)$

- Example of compact subset in $(0,\infty)$?

  - $K = (0,1]$. (? is it compact?)

    K is closed and bounded in X. $\quad X \subset \mathbb{R}$

    - K is closed in X $\because$ $(0,1] = X \cap [0,1]$, with $[0,1]$ closed in $\mathbb{R}$

    - K is bounded in X.

    But Heine–Borel doesn't apply here, since $X \neq \mathbb{R}^n$.

  - K can be covered by $\left\{ B^{(X)}_{\frac{1}{2n}}\left(\frac{1}{n}\right) \right\}_{n\in\mathbb{N}}$.

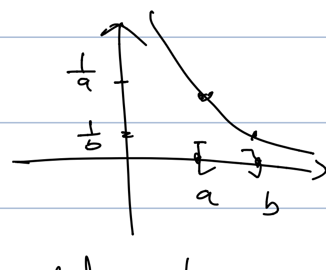
    but it doesn't admit finite subcover. $\Rightarrow$ K is not cpt.

  - K, as a subset of $\mathbb{R}$, is not closed,

    hence K is not compact.

compact subset of $(0,\infty)$ are those compact subset of $\mathbb{R}$, that happens to be in $(0,\infty)$.

e.g. $[a,b] \subset (0,\infty)$.



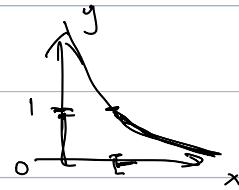
$f([a,b]) = [\frac{1}{b}, \frac{1}{a}] \subset \mathbb{R}$

closed and bounded, hence cpt.

Rmk. preimage of compact set may not be cpt.

if $f: X \to Y$ cont. $F \subset Y$ cpt. then

$f^{-1}(F)$ may not be cpt.



Ex: $f(x) = \frac{1}{x}$ $(x > 0)$. $f^{-1}([0,1])$.

$= [1, \infty)$.

Is $[1,\infty)$ $\underbrace{\text{closed}}_{\text{in } (0,\infty)}$? Yes

But it is not bounded.

Hence, $[1,\infty)$ is not compact in $\mathbb{R}$

$\Rightarrow [1,\infty)$ is not compact.

Ex: $\iota : \underset{X.}{(0,1)} \to \mathbb{R}$

$\iota^{-1}([0,1])$ is not compact.
$\quad = (0,1)$

$(0,1)$ is closed & bounded in X.

but $(0,1)$ is not cpt.