- Continuous functions. Definitions. $f: X \to Y$.
  - 1. ε-δ Language: $\forall p \in X$, $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t.

$$\forall x \in X, \quad d_x(x,p) < \delta \Rightarrow d(f(x), f(p)) < \varepsilon.$$

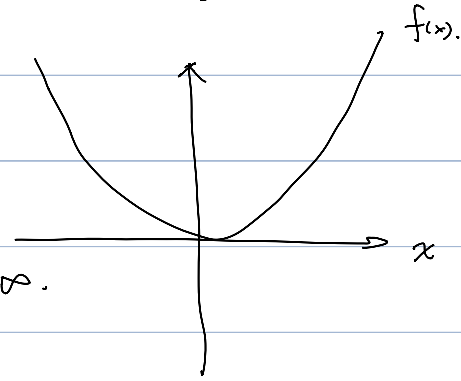
(this constant δ depends on ε, and p).

  - 2. "open set, topological style": $\forall V \subset Y$, open, $f^{-1}(V)$ is open in X
  - 3. "convergent sequences": $\forall$ convergent seq $x_n \to x$, we have $f(x_n) \to f(x)$. in Y

- Def: $f: X \to Y$. fcn between metric spaces.

Suppose for all $\varepsilon > 0$, $\exists \delta > 0$, s.t. $\forall$ $p, q \in X$, with $d_X(p,q) < \delta$. we have $d_Y(f(p), f(q)) < \varepsilon$.

Then, we say f is a uniform continuous function.



Ex: (1). $f(x) = x^2$; $\underset{X}{\mathbb{R}} \to \underset{Y}{\mathbb{R}}$.

$f'(x) = 2x$, $|f'(x)| \to \infty$ as $|x| \to \infty$.

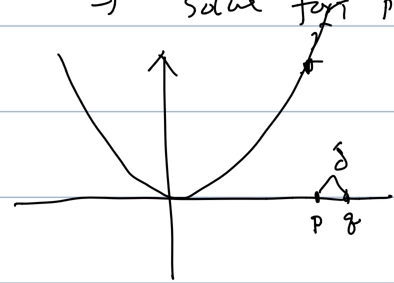
$df(x) = 2x \cdot dx$

Ans: not uniformly continuous. $\forall \delta > 0$, I claim, $\exists p, q \in \mathbb{R}$, $|p - q| < \delta$. s.t. $|f(p) - f(q)| > 1$.

$$|f(p) - f(q)| = |p^2 - q^2| = |(p-q)(p+q)|.$$

we take $p - q = \frac{\delta}{2}$, $p + q = \frac{2}{\delta}$ $\Rightarrow$ solve for p, q.

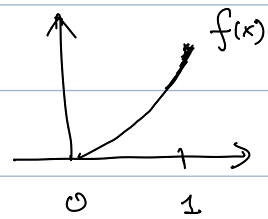
then $|f(p) - f(q)| = 1$.



Intuition: if both p, q are large,

then, even though $|p-q|$ is small. $|p^2-q^2|$ can still be large.

(2) $f(x) = x^2$, $[0,1] \to \mathbb{R}$.



Indeed, this is uniformly continuous.

$\forall \varepsilon > 0$, we can take $\delta = \frac{\varepsilon}{2}$, then

$\forall$ $p, q \in [0,1]$, $|p-q| < \delta$, we have

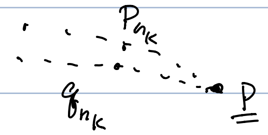
$$|p^2-q^2| = |(p-q)\cdot(p+q)| = |p-q| \cdot |p+q| < \delta \cdot 2 = \varepsilon.$$

✓.

<u>Thm</u>: Suppose $f: X \to Y$ is a cont. fcn, between metric spaces. If X is compact, then f is uniformly continuous.

<u>pf #1</u>: (using sequential compactness). Suppose f is not uniformly cont., then $\exists\, \varepsilon > 0$, s.t. $\forall \delta > 0$, $\exists\, p, q \in X$, $d_X(p,q) < \delta$, s.t. $d_Y(f(p), f(q)) > \varepsilon$. Hence, if we take $\delta = \frac{1}{n}$, for all $n \in \mathbb{N}$, we can get a seq $p_n, q_n \in X$. s.t. $d_X(p_n, q_n) < \delta$, and $d_Y(f(p_n), f(q_n)) > \varepsilon$.

Using seq compactness of X, apply to the seq (p_n), we get a convergent subseq. $(p_{n_k})_k$, say the limit is p.
Then $(q_{n_k})_k$ also converge to p.



By continuity of f, we have

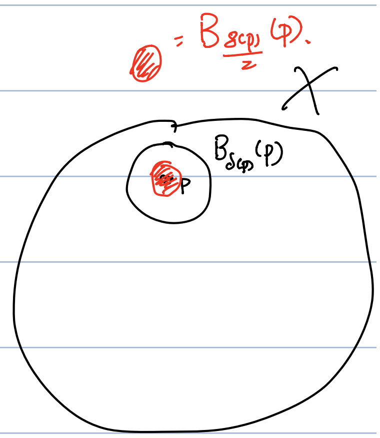
$$\lim_{k\to\infty} f(p_{n_k}) = f(p) = \lim_{k\to\infty} f(q_{n_k})$$

This contradicts, with $d_Y(f(p_{n_k}), f(q_{n_k})) > \varepsilon$ $\forall k$. #.

Pf #2 (using the open cover def'n of compactness).

Given $\varepsilon > 0$. Let $\varepsilon' = \varepsilon/2$. Then, for each $p \in X$, $\exists\, \delta(p) > 0$, s.t. $f(B_{\delta(p)}(p)) \subset B_{\varepsilon'}(f(p))$. Then, we have an open cover of X,

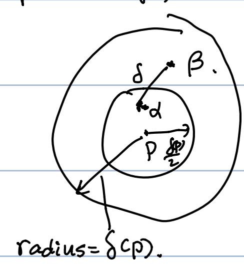
$$X = \bigcup_{p \in X} B_{\delta(p)/2}(p).$$



By compactness of X, we have a finite subset $A \subset X$, s.t.

$$X = \bigcup_{p \in A} B_{\frac{\delta(p)}{2}}(p).$$

Let $\delta = \min\left\{ \frac{\delta(p)}{2} \;\middle|\; p \in A \right\}$. Claim, $\forall\, \alpha, \beta \in X$, s.t. $d_X(\alpha, \beta) < \delta$, we have $d_Y(f(\alpha), f(\beta)) < \varepsilon$.



Pf of claim:

$\because\ \alpha \in X$, $\therefore\ \exists\, p \in A$, s.t. $\alpha \in B_{\frac{\delta(p)}{2}}(p)$.

$$d(\beta, p) < d(\beta, \alpha) + d(\alpha, p) < \delta + \frac{\delta(p)}{2} \leq \frac{\delta(p)}{2} + \frac{\delta(p)}{2} = \delta(p).$$

$\therefore\ \beta \in B_{\delta(p)}(p)$. Now, $\alpha, \beta \in B_{\delta(p)}(p)$.

$\Rightarrow\ d(f(\alpha), f(p)) < \varepsilon'$, $\quad d(f(\beta), f(p)) < \varepsilon'$.

$\Rightarrow\ d(f(\alpha), f(\beta)) \leq \varepsilon' \cdot 2 = \varepsilon$. #.

Prop: If $f : X \to Y$ is uniformly continuous, and $S \subset X$ subset, with induced metric, then the restriction $f|_S : S \to Y$ is uniformly continuous. #.

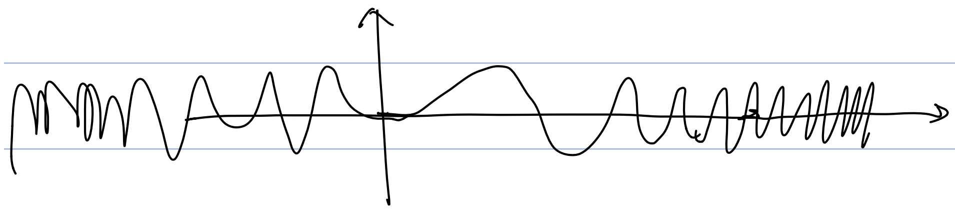
Ex: $f(x) = x^2$ is unif cont. on $(0,1)$.

Pf: $\because$ f is unif cont. on $[0,1]$, since $[0,1]$ is compact.

and $(0,1) \subset [0,1]$.

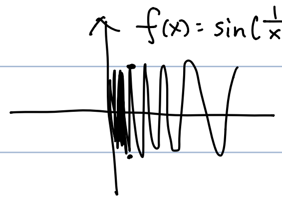
$\therefore$ f is unif ~~co~~ cont. $(0,1)$.

Ex: • $f(x) = \sin(x^2)$: $\mathbb{R} \to \mathbb{R}$ is NOT unif cont.



• $f(x) = \frac{1}{x}$, $(0,\infty) \to \mathbb{R}$. is NOT unif cont.

similarly. $f(x) = \underline{\sin(\frac{1}{x})}$, $(0,\infty) \to \mathbb{R}$. is NOT unif cont.



if $p = \frac{1}{2\pi(n+\frac{1}{4})}$, $q = \frac{1}{2\pi(n-\frac{1}{4})}$,

$f(p) = \sin(2\pi(n+\frac{1}{4})) = \sin(2\pi n + \frac{\pi}{2}) = 1$

$f(q) = \sin(2\pi n(n-\frac{1}{4})) = \sin(2\pi n - \frac{\pi}{2}) = -1$.

$|f(p) - f(q)| = 2.$, while $|p-q|$ can be made small..

$\equiv$.

Connectedness:

• Let X be a topological space. (i.e. a set endowed with a collection of open subset...)

We say X is <u>connected</u>, if $\forall S \subset X$,

if S is both open and closed, then S has to be either

X or ϕ.

$X =$ 

Ex: $X = [0,1] \cup [2,3]$.

X is not connected, because $[0,1] \subset X$ is both open and closed. subset of X.

to show $[0,1]$ is open, just need.

$$[0,1] = \bigcup_{p \in [0,1]} B^{(X)}_{\frac{1}{2}}(p) \qquad B^X_r(p) = \{ q \in X \mid d(q,r) < r \}$$

open.

$[0,1]$ is a union of open sets. hence is open.

Prop: X is connected $\iff$ If $X = U \amalg V$, and U and V are both open, then one of U, V is empty set.

For $U, V \subset X$, if $U \cap V = \phi$, then we write $U \cup V$ as $U \sqcup V$, say it is the disjoint union.

Pf $\Rightarrow$ If X conn, and $X = U \amalg V$, U, V open. then $U^c = V$ which is open. hence U is both open and closed, hence. $U = X$ or ϕ.

Thm: If $f: X \to Y$ is continuous, if $E \subset X$ is connected, then $f(E)$ is connected.

Pf: let $h = f|_E$, and consider $h: E \to f(E)$.

If $f(E)$ is not connected, then $\exists\, U, V$ open in $f(E)$, s.t. $U \cap V = \phi$, $f(E) = U \amalg V$. Consider preimage of h.

$$E = h^{-1}(U) \amalg h^{-1}(V).$$

E is a union of non-empty disjoint open subsets (of E).

hence E is not connected.

#

Recall, induced topology on a subset $S \subset X$.

An subset $U \subset S$ is open in S if and only if, $\exists\, \tilde{U} \subset X$ open in X. such that

$$U = \tilde{U} \cap S.$$

- Apply it to functions: $f: \mathbb{R} \to \mathbb{R}$.

- Prop: $[0,1] \subset \mathbb{R}$ is a connected subset.

Pf: we prove by contradiction. Say $[0,1] = A \amalg B$, A, B are non-empty. open subsets in $[0,1]$. i.e. $\exists\, \tilde{A}, \tilde{B} \subset \mathbb{R}$ open. s.t.

$A = \tilde{A} \cap [0,1]$, $B = \tilde{B} \cap [0,1]$.

If A, B are open in $[0,1]$, then

$$A = [0,1] \setminus B. \qquad B = [0,1] \setminus B.$$

then A, B are also closed in $[0,1]$.

$\because$ $[0,1]$ is compact, and closed subset of compact space is compact

$\therefore$ A, B are compact.

$\sup(A) \in A$, $\inf(A) \in A$. $\sup(B) \in B$, $\inf(B) \in B$.

(next time, finish the proof. Goal: try to find a point that's the limit point of both A and B.) #.

$\sup(A) \neq \sup(B)$. say $\sup(B) < \sup(A)$.

then $\exists\, p_n \in A$, $p_n > \sup(B)$, $p_n \to \sup(B)$.

~~that~~ $\because$ $\lim p_n$ exists, $\therefore$ $\lim p_n \in A$ by closedness of A.

$\therefore$ $\sup(B) \in A$.