Rudin Ch 4

## Limit of Functions

**Def** let X & Y be metric spaces; suppose $E \subset X$, f maps E into Y, and p is a limit point of E. We write $f(x) \to q$ as $x \to p$ or $\lim_{x \to p} f(x) = q$

if $\exists\, q \in Y$ st for every $\varepsilon > 0$ $\exists\, \delta > 0$ st
$d_Y(f(x), q) < \varepsilon$
$\forall\, x \in E$ where $0 < d_X(x, p) < \delta$

**Theorem** $\lim_{x \to p} f(x) = q$ iff $\lim_{n \to \infty} f(p_n) = q$ for every seq $\{p_n\}$ in E st. $p_n \neq p$ & $\lim p_n = p$

**Corl** If f has limit at p then p is unique

**Theorem** $E \subset X$, metric space, p is lim pt of E, f & g are complex func on E,
& $\lim_{x \to p} f(x) = A$ $\lim_{x \to p} g(x) = B$

**Then**
- $\lim_{x \to p} (f+g)(x) = A + B$
- $\lim (fg)(x) = AB$
- $\lim (f/g)(x) = A/B$ if $B \neq 0$

## Continous Functions

**Def** X, Y are metric spaces, $E \subset X$, $p \in E$, & f maps E into Y.
Then f is said to be continous at p if for every $\varepsilon > 0$
$\exists\, \delta > 0$ s.t. $d_Y(f(x), f(p)) < \varepsilon$
for all $x \in E$ when $d_X(x, p) < \delta$

**Thrm** f is cont at p iff $\lim_{x \to p} f(x) = f(p)$

**Thrm** Suppose $X, Y, Z$ are metric spaces, $E \subset X$,
$f: E \to Y$, $g: \text{range}(f) \to Z$   $h: E \to Z$   $h(x) = g(f(x))$   $(x \in E)$
If $f$ is cont at pt $p \in E$ & if $g$ is cont at the pt $f(p)$
then $h$ is cont at $p$.

**Thrm** A mapping $f$ of a metric space $X$ into a metric space $Y$ is continuous on $X$ **iff** $f^{-1}(V)$ is open in $X$ $\forall$ open sets $V$ in $Y$

**Corl** A map $f: X \to Y$ is cont **iff**
$f^{-1}(C)$ is closed in $X$ $\forall$ closed set $C$ in $Y$.

**Theorem** $f_1, \ldots, f_k$ real func on $X$, & let $f: X \to \mathbb{R}^k$
$f(x) = (f_1(x), \ldots, f_k(x))$   $(x \in X)$
**Then** $f$ is cont iff $f_i$ is cont $\forall i$

## Continuity & Compactness

**Def** A map $f: E \to \mathbb{R}^k$ is **bounded** if $\exists$ real # $M$ s.t.
$|f(x)| \leq M \quad \forall x \in E$

**Thrm** $f$: compact $X \to Y$ $\Rightarrow$ $f(X)$ is compact

**Thrm** $f$ cont map: comp $X \to \mathbb{R}^k$ $\Rightarrow$ $f(X)$ is closed & bounded $\Rightarrow$ $f$ is bound

**Thrm** $f$ cont real func on comp metric space $X$
and $M = \sup_{p \in X} f(x)$   $m = \inf_{p \in X} f(p)$
$\Rightarrow$ $\exists$ pts $p, q \in X$
s.t $f(p) = M$ & $f(q) = m$

**Thrm** $f$ 1:1 cont map: comp $X \to Y$. $\Rightarrow$ $f^{-1}$ defined on $Y$ by
$f^{-1}(f(x)) = x$   $(x \in X)$

is a continuous mapping of $Y$ onto $X$

<u>Def</u> let $f: X \to Y$. $f$ is <u>uniformly continuous</u> on $X$ if
for every $\varepsilon > 0$, $\exists \delta > 0$
s.t. $d_Y(f(p), f(q)) < \varepsilon$
$\forall$ $p$ & $q$ in $X$ for which $d_X(p, q) < \delta$

<u>Thrm</u> let $f$ be cont map of compact metric space $X \to$ met sp $Y$ $\Rightarrow$ $f$ is uniformly cont on $X$

<u>Theorem</u> let $E$ be noncompact set $E$ in $\mathbb{R}$.
- $\exists$ a cont func on $E$ which is not bounded
- There exist cont. & bounded func on $E$ with no max

↳ If $E$ is bounded
- $\exists$ a cont func on $E$ which is not uniformly cont.

<u>Continuity & Connectedness</u>
<u>Theorem</u> If $f: X \to Y$, $E$ is <u>connected</u> subset of $X$,
$\Rightarrow$ $f(E)$ is connected

<u>Theorem</u> let $f$ be a cont real func. on the interval $[a,b]$. If $f(a) < f(b)$ & if $c$ is a number s.t. $f(a) < c < f(b)$
(squeeze thrm

## Discontinuities

- $\lim f(t)$ exists iff $f(x+) = f(x-) = \lim f(t)$

**Simple Discont**

1) $f(x-) \neq f(x+)$ 2) $f(x-) = f(x+) \neq f(x)$

## Monotonic Functions

**Def** $f$ on $(a,b)$. $f$ is said to be mont. inc. on $(a,b)$ if $a < x < y < b$ implies $f(x) \leq f(y)$

↳ reverse for mont dec func

**Thrm** $f$ mont inc $(a,b) \Rightarrow f(x+)$ & $f(x-)$ exist $\forall x \in (a,b)$

↳ $\sup_{(a,x)} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{(x,b)} f(t)$

~~Thrm~~

Monotonic func have no non simple disconts

**Thrm** Let $f$ be monotonic $\overset{(a,b)}{\Rightarrow}$ set. of pts in $(a,b)$ where $f$ is discont is at most countable.