Feb 16 Metric Space and Topology

Metric Spaces

↪ S: set
$d: S \times S \to \mathbb{R}_{\geq 0}$ distance metric

Distance Metric
- $d(x,y) \geq 0$
- $d(x,y) = d(y,x)$
- $d(x,y) + d(y,z) \geq d(x,z)$

Examples:
$|x-y|$
$\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$
$L^p = \left[\sum_{i=1}^{n} |x_i - y_i|^p\right]^{\frac{1}{p}}$

$(S_n)_n$ is Cauchy if $\forall \varepsilon > 0$, $\exists N > 0$ s.t. $\forall n, m > N$

$$d(S_n, S_m) < \varepsilon$$

most def in metric space are the same but with d as distance metric

convergence

(S_n) converges to s if $\forall \varepsilon > 0, \exists N > 0$
s.t. $d(S_n, s) < \varepsilon$ $\forall n > N$

$(S_n)_n$ converges $\Rightarrow$ cauchy

Complete Space

Complete $\Rightarrow$ every cauchy sequence is convergent
or
Every Cauchy sequence has a limit in S

Comparable Metrics

$\exists C_1, C_2 > 0$ s.t.

$$C_1 \cdot d_p(x,y) < d(x,y) < C_2 \cdot d_p(x,y$$

For $\mathbb{R}$

bounded seq $\Rightarrow$ convergent subseq

For $\mathbb{R}^n$

$(S_n)_n$ convergent $\Leftrightarrow$ $(S_n^{(i)})_n$ is convergent $\forall i$

Bolzano–Weierstrass:

Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence

$\mathbb{R}^n$ is complete

Topology

Topology on a set S is a collection of open subsets s.t.

- S, $\emptyset$ are open
- if the union of a collection of open subsets is open
- if the intersection of a finite collection of open subsets is open

For a metric space

$B_r(p) = \{ x \in S \mid d(p,x) < r \}$

$B_r(p)$ is open

$\hookrightarrow$ collection of $B_r(p)$ can create a topology

Induced Topology

We equip S with the induced topology $E \subset S \subset X$ where E is open in S iff there exists an open subset $\tilde{E} \subset X$ s.t. $E = S \cap \hat{E}$

^if E is in S and X and open in X

Ex: $(0,1]$ closed in $(0,\infty) := X$ since $[0,1]$ is closed in $\mathbb{R}$ $(0,1] = X \cap [0,1]$

Induced Distance Function

If (S,d) is a metric space and $A \subset S$, then (A,d) is a metric space