Feb 2

Cauchy Sequence

Cauchy if $\forall \varepsilon > 0, \exists N > 0$ st.

$|a_n - a_m| < \varepsilon \quad \forall n,m > N$

any two points in the tail are $< \varepsilon$ apart

THM: a_n is a cauchy seq $\iff$ a_n converges $\iff$ $\limsup a_n = \liminf a_n$

Most of the lecture is just proving

- <u>a_n converges $\Rightarrow$ a_n cauchy sequence</u>

  $\lim a_n = \alpha$ by convergence

  $|a_n - \alpha| < \frac{\varepsilon}{2}$ by def of limit

  $|a_m - \alpha| < \frac{\varepsilon}{2}$ for some $m > N$

  $|a_n - \alpha| + |\alpha - a_m| < \varepsilon$ by triangle inequality

  $|a_n - a_m| < \varepsilon$ ■

- <u>lemma:</u> $\forall \varepsilon > 0, \exists N > 0$, s.t. $\forall n > N$ $\quad a_n < \limsup a_n + \varepsilon$

  limsup is the sup of the tail with an epsilon of room

  $\limsup a_n = \lim S_N = \alpha$

  $|S_N - \alpha| < \varepsilon$

  $\alpha - \varepsilon < S_N < \alpha + \varepsilon$

  $a_n \leq S_N$ since $S_N = \{\sup a_n \quad n \geq N\}$

  $a_n < \alpha + \varepsilon$

  $a_n < \limsup a_n + \varepsilon$ ■

- $\forall \varepsilon > 0, \exists N > 0$ st $\forall n > N$ $\quad a_n > \liminf a_n + \varepsilon$, proof similar to

- <u>$\limsup a_n = \liminf a_n \Rightarrow$ convergence</u>

  $\limsup a_n = \liminf a_n \Rightarrow \lim a_n$ exists and $= \liminf a_n$ from lemmas

  $\lim a_n$ exists $\Rightarrow$ convergence

- <u>a_n convergence $\Rightarrow \limsup a_n = \liminf a_n$</u>

  convergence $\Rightarrow$ lim exists $\Rightarrow \lim a_n = \liminf a_n = \limsup a_n$

- a_n is cauchy $\Rightarrow$ a_n converges

  $|a_n - a_m| < \varepsilon$

  for some fixed a_n , a_m is bounded

  thus (a_n) is bounded

cauchy sequence