Feb 23 Open and Closed Sets

Open Set

- $\forall p \in U \; \exists \delta > 0$ st $B_\delta(p) \subset U$
- $S, \emptyset$ are open
- arbitrary union of open subsets is open
- EX: (a,b), (a,∞), $(-\infty,\infty)$, $\mathbb{R}$

Closed Set

- E is closed iff E^c is open
- $S, \emptyset$ are closed
- arbitrary intersection of closed subsets is closed
- $[a,b]$, $[a,\infty)$, $(-\infty, a]$, $\{a\}$, $\mathbb{R}$

Closure

intersection of all closed subsets of S that contain F

For $E \subset S$, the closure of $\bar{E} = \bigcap \{F \mid F \subset S \text{ closed set}, F \supset E\}$

Interior

$$E^\circ = \bigcup \{u \mid u \subset S \text{ open}, u \subset E\}$$
$$= \{p \in E \mid \exists \delta > 0 \; B_\delta(p) \subset E\}$$

union of all open subsets of E

Boundary

$\bar{E} \setminus E^\circ$ whatever is in the closure and not in the interior

E some subset of S
$\bar{E}$ closure
E° interior
E' limit points in E

$$\bar{E} = E \cup E'$$

Limit Point

Every $B_r(p)$ $\forall r > 0$ contains a point $q \neq p$ such that $q \in E$

Limit Point
Every $B_r(p)$ $\forall r > 0$ contains a point $q \neq p$ such that $q \in E$

Isolated Point
Not a limit point

Closed Set
Every limit point of $E \subset E$ / $\forall$ convergent sequence (x_n) in S $x = \lim x_n$
if x_n are in E, then $x \subset E$

Interior Point
$\exists B_r(p)$ s.t. $B_r(p) \subset E$

Open Set
Every point in E, $\exists B_r(p)$ s.t. $B_r(p) \subset E$

Complement
$E^c = \forall p \notin E$

Perfect Set
E is closed and $\forall p \in E$ are limit points

Dense Set
$\forall p \in X$ is a limit point of E or a point of E
$\Rightarrow$ E is dense in X

Open Cover
collection $\{G_\alpha\}$ of open subsets of X s.t. $E \subset \bigcup_\alpha G_\alpha$

Compact Set
Every open cover of K contains a finite subcover
- every finite set is compact
- $K \subset Y \subset X$, K is compact in $X \Leftrightarrow K$ is compact in Y
- independent of ambient space

Sequentially Compact Set
$\forall$ sequence (x_n) in K, $\exists x \in K$ and a subseq $(x_{n_k})_k$
For any sequence, $\exists$ a subsequence that converges

THM

$E \subset S$ is closed $\iff$ $\forall$ convergent sequences in S, where (x_n) are in E, x is in E
$\lim x_n = x$

Heine-Borel

for R^n, $K \subset R^n$ is compact $\Leftrightarrow$ K is closed and bounded $\Leftrightarrow$ infinite subset of E has a limit point in E

n-cells in R^n are compact

↳ n-cell is $[a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$
suppose there is a non-compact cell → prove by contradiction