Sequence and Limits

Sequence

a collection of numbers indexed by $n \in \mathbb{N}$ a function $\mathbb{N} \to \mathbb{R}$

$a_1, a_2, a_3 \dots$ $a_n \in \mathbb{R}$ $\forall n \in \mathbb{N}$

Limit

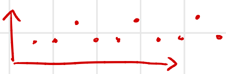
$$\lim_{n \to \infty} a_n = \alpha$$ short version

long version

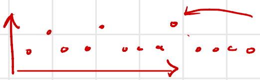
consider some $a_1, a_2, \dots a_n$ sequence, a_i converges to $\alpha \in \mathbb{R}$ if $\forall \varepsilon > 0$, we find $N > 0$ s.t. for any $n > N$ $n \in \mathbb{N}$

$$|a_n - \alpha| < \varepsilon$$

Adverserial Examples → No limit exists



and so on



increasing gap between

PROVING THE LIMIT EXISTS WITH DEF

- ↳ "guess" the limit
- ↳ set $f(n) < \varepsilon$ solve for n
- ↳ use $N =$ solution from ↑
- ↳ show that definition of lim is satisfied with N we found

OTHER OPTION
↳ use limit rules for add, mul, div, etc

Ex: $\lim_{n \to \infty} \frac{1}{n^2} = 0$

$$\frac{1}{n^2} < \varepsilon$$
$$\frac{1}{\varepsilon} < n^2$$
$$\frac{1}{\sqrt{\varepsilon}} < n$$
$$N = \frac{1}{\sqrt{\varepsilon}}$$

Lets take $N = \frac{1}{\sqrt{\varepsilon}}$, then any $n > N$, we have

$a_n < \varepsilon$, thus $|a_n - 0| < \varepsilon$ ∎

TOOLS TO COMPUTE LIMITS

THM: All convergent sequences are bounded

If $\lim_{n\to\infty} a_n = \alpha$, then $\exists M>0$ s.t. $|a_n| \leq M$ $\forall n \in \mathbb{N}$

THM: If $\lim_{n\to\infty} a_n = \alpha$, and $\underbrace{a_n \neq 0, \ \forall n \in \mathbb{N},}_{\text{need both statements}}$ and $\alpha \neq 0$
then, $\left(\frac{1}{a_n}\right)$ is a bounded sequence

MANY USEFUL THMS

Let $a_n \to \alpha$, $b_n \to \beta$ as $n \to \infty$

1) For any $k \in \mathbb{R}$, $\lim_{n\to\infty} k a_n = k \cdot \lim_{n\to\infty} a_n = k\alpha$

2) $\lim_{n\to\infty} (a_n + b_n) = \lim a_n + \lim b_n = \alpha + \beta$

3) $\lim_{n\to\infty} a_n b_n = \lim a_n \cdot \lim b_n = \alpha\beta$

4) If $b_n \neq 0$ $\forall n$ and $\beta \neq 0$, then $\lim \left(\frac{a_n}{b_n}\right) = \frac{\lim a_n}{\lim b_n}$