January 28

Limit Proofs

Ex: $\lim\left(\frac{1}{b_n}\right) = \frac{1}{\lim b_n}$

Find some $N>0$ s.t. $\forall \varepsilon>0$ $\left|\frac{1}{b_n} - \frac{1}{\beta}\right| < \varepsilon$ $\forall n>N$

$$\left|\frac{\beta - b_n}{b_n \beta}\right| < \varepsilon$$

Since $\frac{1}{b_n}$ is a bounded sequence, $\left|\frac{1}{b_n}\right| < C$ $\forall n$

$$\left|\frac{1}{b_n}\left(\frac{\beta - b_n}{\beta}\right)\right| < \left|\frac{\beta - b_n}{\beta}\right| \cdot C \quad \forall n$$

$\left|\frac{\beta - b_n}{\beta}\right| \cdot C < \varepsilon$ would satisfy condition for limit

$$|\beta - b_n| < \frac{\varepsilon}{C} \cdot |\beta| \quad \forall n$$

since $b_n \to \beta$, we know $|\beta - b_n| < \varepsilon_1$, thus we can satisfy

Ex: $a_n = \frac{1}{n^p}$ $p>0$ $\lim a_n = 0$

We need some $N>0$, $\forall \varepsilon > 0$ s.t.

$$\frac{1}{n^p} < \varepsilon \quad \forall n > N$$

$$\frac{1}{\varepsilon} < n^p$$

$$\left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}} < n$$

thus, we take $N = \left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}}$ and thus we can show the orignal statement

Most lim proofs follow a similar format

Limits at infinity

Def: Let a_n be a seq $a_n \in \mathbb{R}$
$\lim a_n = \infty$ if $\forall M > 0, \exists N > 0$ s.t. $a_n > M \quad \forall n > N$

Monotone sequences

(a_n) is increasing if $a_{n+1} \geq a_n$

(a_n) is decreasing if $a_{n+1} \leq a_n$

THM: If a_n is increasing and bounded, then (a_n) is convergent

THM: All bounded and monotone seq are convergent

(a_n) is an unbounded increasing sequence, then $\lim s_n = \infty$

(a_n) is an unbounded decreasing sequence, then $\lim s_n = -\infty$

$\lim a_n$ is always meaningful for monotone sequences
aka for seq that dont oscilate forever

lim sup, lim inf

$\limsup a_n = \lim_{N \to \infty} S_n$ where $S_n = \sup\{a_n \mid n \geq N\}$ sup of the tail part of a_n

$\liminf a_n = \lim_{N \to \infty} S_n$ where $S_n = \inf\{a_n \mid n \geq N\}$

THM: If $\lim a_n$ is defined, then $\limsup a_n = \lim a_n = \liminf a_n$

THM: If $\limsup a_n = \liminf a_n$, then $\lim a_n$ is defined

S_n is a decreasing sequence, thus $\limsup a_n$ is always defined, and equivalently for $\liminf a_n$

Cauchy Sequence

Cauchy if $\forall \varepsilon > 0, \exists N > 0$ st.
$|a_n - a_m| < \varepsilon \quad \forall n, m > N$

THM: a_n is a cauchy seq $\iff$ a_n converges $\iff$ $\limsup a_n = \liminf a_n$