March 2nd — Continuity of Functions

Injective

$\forall x, y \in A$ and $x \neq y$, then $f(x) \neq f(y)$

Surjective

$\forall \beta \in B \; \exists \alpha \in A$ s.t. $f(\alpha) = \beta$

Continuity

1) $f: X \to Y$, for every $\epsilon > 0 \; \exists \, \delta > 0$ s.t.
$$d_Y(f(x), f(p)) < \epsilon$$
$$\forall x \in E \quad d_X(x, p) < \delta$$

2) continuity $\Leftrightarrow$ $f^{-1}(V)$ is open in X for every $V \subset Y$ that is open

$\Leftrightarrow$ $f^{-1}(C)$ is closed in X for every $C \subset Y$ that is closed

3) continuity $\Leftrightarrow$ $\forall$ convergent seq $x_a \to x$ in X, we have $f(x_a) \to f(x)$ in Y

limit of a function

let X, Y be metric spaces, $E \subset X$ is a subset, and $f: E \to Y$

let p be a limit point of E, then

$\lim_{x \to p} f(x) = q$ if $q \in Y$ s.t. $\forall \epsilon > 0, \exists \, \delta > 0$ s.t.
$$f(B_\delta^*(p) \cap E) \subset B_\delta(q)$$

OR

$$\forall x \in E \text{ s.t. } 0 < d_X(x, p) < \delta \quad d_Y(f(x), q) < \epsilon$$

$\lim_{x \to p} f(p_n) = q \Leftrightarrow \forall$ convergent seq $p_n \to p$ with $p_n \in E$
$$\lim_{n \to \infty} f(p_n) = q$$

THM: Let $f, g : X \to \mathbb{R}$ $\lim_{x \to p} f(x) = A$ $\lim_{x \to p} g(x) = B$

↳ $\lim_{x \to p} (f+g)(x) = A + B$

↳ $\lim_{x \to p} (f \cdot g)(x) = A \cdot B$

↳ $\lim_{x \to p} (f/g)(x) = A/B$ if $B \neq 0$, $g(x) \neq 0$ $\forall x \in X$

If f, g are continuous, $f+g$, $f \cdot g$, f/g are continuous

also $f: X \to Y$ $g: Y \to Z$ continuous, then $g \circ f : X \to Z$ is continuous

<u>THM</u> $f: X \to \mathbb{R}^n$ with $f(x) = (f_1(x), f_2(x) \ldots)$

↳ f is continuous $\Leftrightarrow$ $f_i : X \to \mathbb{R}$ are continuous $\forall i = 1 \ldots n$

<u>Compactness</u>

- $f: X \to Y$ continuous and $E \subset X$ compact, then $f(E) \subset Y$ is compact

← preimage of compact set may not be compact

Ex $(0, \infty) \to \mathbb{R}$

$f^{-1}([0,1])$

- If $f: X \to \mathbb{R}$ continuous, and $E \subset X$ is compact, then $\exists p, q \in E$ s.t. $f(p) = \sup(f(E))$ and $f(q) = \inf(f(E))$