Notes for MATH 104

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Contents

1 Topology

1.1 Basic Definitions

Definition 1.1. A set X is said to be a **metric space** if with any two points p and q of X there is associated a real number $d(p, q)$, called the distance from p to q, such that

$$\begin{aligned}
&(a) \quad d(p,q) > 0 \; if \; p \neq q; \; d(p,p) = 0; \\
&(b) \quad d(p,q) = d(q,p); \\
&(c) \quad d(p,q) \leq d(p,r) + d(r,q), \; \forall \, r \in X.
\end{aligned}$$

where (c) is called the **Triangle Equation**.

Let X be a metric space. All points and sets discussed below are understood to be elements and subsets of

Definition 1.2. A **neighborhood** of a point p is a set $N_r(p) = \{x \in X \mid d(x,p) < r\}$. r is called the radius of $N_r(p)$.

Definition 1.3. A point p is **interior** to (or is an interior point of) E if $\exists \, r > 0, \; s.t. \; N_r(p) \subseteq E$.

Definition 1.4. A set E is **open** if every points in E is interior to E.

Definition 1.5. A point p is a **limit point** of E if $\forall \, r > 0, \; s.t. \; \exists \, x \neq p, \; s.t. \; x \in N_r(p) \cap E$

Definition 1.6. A set E is **closed** if every limit point of E is a point of E.

Remark. Another definition of closed set in Ross: A set E is closed if its complement E^c (denoted by $X \backslash E$ in Ross) is open.

Definition 1.7. The **complement** of E (denoted by E^c in Rudin) is the set of all points $p \in X$ such that $p \notin E$. i.e. $E^c = \{x \in X \mid x \notin E\}$.

Definition 1.8. A set E is **bounded** if $\exists \; M \in \mathbb{R} \; and \; q \in X, \; s.t. \; \forall \; p \in E, \; d(p,q) < M$.

Remark. Another definition of bounded sets: A set E is bounded if $\exists \; M \in \mathbb{R}, \; s.t. \; \forall \, p, q \in E, \; d(p,q) < M$.

Definition 1.9. A set E is **dense** if every point of X is a limit point of E or a point of E. i.e. $X = \overline{E}$.

Definition 1.10. The **closure** of a set E (denoted by $\overline{E}$) is $\overline{E} = E \cap E'$. (We use E' to denote the set of limit points of E)

Remark. Another definition of closure in Ross: The closure E^- of a set E is the intersection of all closed sets containing E.

Definition 1.11. The **boundary** of E is the set $E^- \backslash E^o$, where E^o denotes the set of interior points of E.

Definition 1.12. A set E is **perfect** if E is closed and every point of E is a limit point of E. i.e. $E = E'$.

1.2 Basic Theorems

Theorem 1.1. Every neighborhood is open.

Theorem 1.2. If $p \in E'$, then $\forall\ r > 0,\ N_r(p) \cap E$ is infinite.

Corollary 1.2.1. A finite set has no limit points.

Theorem 1.3. The union of any collection of open sets is open.

Corollary 1.3.1. The intersection of any collection of closed sets are closed.

Theorem 1.4. The intersection of finitely many open sets is open.

Corollary 1.4.1. The union of finitely many closed sets are closed.

Theorem 1.5. The set E is closed iff. $E = \overline{E}$.

Theorem 1.6. The set E is closed iff. it contains the limit of every convergent sequence of points in E.

Theorem 1.7. An element is in $\overline{E}$ iff. it is the limit of some convergent sequence of points in E.

Theorem 1.8. $x \in \partial E$ iff. $x \in \overline{E} \cap \overline{E^c}$, where ∂E means the boundary of E.

Theorem 1.9. Suppose $Y \subset X$. A subset E is open relative to Y iff. $E = Y \cap G$ for some open subset G of X.

1.3 Compactness

Definition 1.13. An **open cover** of a set E in a metric space X is a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \cup_\alpha G_\alpha$.

Definition 1.14. A subset K of a metric space X is said to be **compact** if every open cover of K contains a finite subcover.

Theorem 1.10. Suppose $K \subset Y \subset X$. Then K is compact relative to X iff. K is compact relative to Y.

Remark. Comparing with **Theorem 1.9**, we can see that compactness can be seen as a property of metric spaces.

Theorem 1.11. Compact subsets of metric spaces are closed.

Theorem 1.12. Closed subsets of compact sets are compact.

Corollary 1.12.1. If F is closed and K is compact, then $F \cap K$ is compact.

Theorem 1.13. If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that in the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\cup K_\alpha$ is nonempty.

Corollary 1.13.1. If $\{K_\alpha\}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$, then $\cap_1^\infty K_n$ is not empty.

Corollary 1.13.2. (Ross 13.10 Theorem) Let (F_n) be a decreasing sequence [i.e. $F_n \supset F_{n+1}$] of closed bounded nonempty sets in $\mathbb{R}^k$. Then $F = \cap_{n=1}^\infty F_n$ is also closed bounded and nonempty.

Theorem 1.14. Suppose E is an infinite subset of a set K. Then E has a limit point in K iff. K is compact.

Remark. $\Longrightarrow$ can be found in 2.37 Theorem of Rudin. $\Longleftarrow$ can be found in Excercise 26 of Rudin Chapter 2.

Theorem 1.15. Every k-cell is compact.

Theorem 1.16. If a set E in $\mathbb{R}^k$ has one of the following three properties, then it has the other two:
(a) E is closed and bounded
(b) E is compact
(c) Every infinite subset of E has a limit point in E

Corollary 1.16.1. Every bounded infinite subset of R^k has a limit point in $\mathbb{R}^k$.

1.4 Connectedness

Definition 1.15. The sets A and B are said to be separated if $A \cap \overline{B} = \emptyset$ and $B \cap \overline{A} = \emptyset$.

Definition 1.16. A set $E \subset X$ is said to be connected if E is not a union of two nonempty separated sets.

Remark. Another definition on Ross about disconnected sets are rather complicated with the concept of nonempty relatively-open subsets. So I leave it out.

Theorem 1.17. A subset E of the real line $\mathbb{R}^1$ is connected iff. it has the following property: If $x \in E,\ y \in E$, and $x < z < y$ then $z \in E$.

Remark. More contents about compactness and connected sets are associated with continuity. See the following sections.

2 Continuity

2.1 Limits of Functions

Definition 2.1. Let X and Y be metric spaces; suppose $E \subset X$, f maps E into Y, and p is a limit point of E. We write $f(x) \to q$ as $x \to p$, or

$$\lim_{x \to p} f(x) = q$$

if there is a point $q \in Y$ with the following property: For every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(d(x), q) < \epsilon$$

for all points $x \in E$ for which

$$0 < d_X(x, p) < \delta$$

Remark. This is called the two-sided limit by Ross. In Ross, there is $\lim_{x \to p^E} f(x) = q$ iff. $\lim_{x \to p} f(x) = q$ and $f(x) = q$. Then Ross defined two-sided and one-sided limits based on this.

Theorem 2.1. $\lim_{x \to p} f(x) = q$ iff. $\lim_{n \to \infty} p_n = q$ for every sequence $\{p_n\}$ such that $p_n \neq p$ and $\lim_{n \to \infty} p_n = p$.

Corollary 2.1.1. If f has a limit at p, this limit is unique.

Theorem 2.2. Suppose $lim_{x \to p} f(x) = A$ and $lim_{x \to p} g(x) = B$. Then

$$(a) \quad \lim_{x \to p} (f + g)(x) = A + B;$$

$$(b) \quad \lim_{x \to p} (fg)(x) = AB;$$

$$(c) \quad \lim_{x \to p} (f/g)(x) = A/B, \; if B \neq 0.$$

2.2 Continuous Functions

Definition 2.2. Let X and Y be metric spaces; suppose $E \subset X$, f maps E into Y, and $p \in E$. Then f is said to be **continuous** at p if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(x), f(p)) < \epsilon$$

for all points $x \in E$ for which $d_X(x, p) < \delta$.

Remark. Another definition of continuous functions in Ross: f is continuous at p iff. for every sequence (x_n) of points in E converging to p, $\lim_{n\to\infty} f(x_n) = f(p)$. This definition can be convenient in some proofs.

Theorem 2.3. Assume $p \in E$ is also a limit point of E, then f is continuous iff. $\lim_{x\to p} f(x) = f(p)$.

Theorem 2.4. If f is a real-valued function continuous at x_0, then $|f|$ and kf is also continuous at x_0.

Theorem 2.5. If f, g are real-valued functions continuous at x_0, then $f + g$, fg, $\max(f, g)$, $\min(f, g)$ is also continuous at x_0, and f/g is continuous at x_0 if $g(x_0) \neq 0$.

Theorem 2.6. If f is continuous at x_0 and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.

Theorem 2.7. A mapping f of a metric space X into a metric space Y is continuous on X iff. $f^{-1}(V)$ is open in X for every open set V in Y.

Corollary 2.7.1. A mapping f of a metric space X into a metric space Y is continuous on X iff. $f^{-1}(C)$ is closed in X for every closed set C in Y.

Definition 2.3. $f : X \to Y$. f is **uniformly continuous** on X if for every $\epsilon > 0$ there exists $\delta > 0$ such that
$p, q \in X$ and $d_X(p, q) < \delta$ implies $d_Y(f(p), f(q)) < \epsilon$.

2.3 Continuity and Compactness

Theorem 2.8. Suppose f is a continuous mapping of a compact metric space Y. Then f(x) is compact.

Corollary 2.8.1. If f is a continuous mapping of a compact metric space X into $\mathbb{R}^k$ then $f(x)$ is closed and bounded. Thus f is bounded.

Corollary 2.8.2. Suppose f is a continuous real function on a compact metric space X, then there exists points $p, q \in X$ such that $f(p) = \sup_{x\in X} f(x)$ and $f(q) = \inf_{x\in X} f(x)$. i.e. f attains its maximum and minimum on X.

Theorem 2.9. Suppose f is a continuous 1-1 mapping of a compact metric space X onto a metric space Y. Then the inverse mapping f^{-1} is a continuous mapping of Y onto X.

Theorem 2.10. Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X.

2.4 Continuity and Connectedness

Theorem 2.11. If $f : X \to Y$ is continuous, and if E is a connected subset of X, then $f(E)$ is connected.

Corollary 2.11.1. (Intermediate Value Theorem) Let f be a continuous real function on the interval $[a, b]$. If $f(a) < c < f(b)$, then there exists a point $x \in (a, b)$ such that $f(x) = c$.

Corollary 2.11.2. If f is a continuous real-valued function on an interval I, then the set $f(I) = \{f(x) \mid x \in I\}$ is also an interval or a single point.

2.5 Discontinuities

Definition 2.4. Let f be defined on (a, b). We write $f(x+) = q$ (Rudin) or $\lim_{t \to x^+} f(t) = q$ (Ross) if $f(t_n) \to q$ as $n \to \infty$ for every sequence (t_n) in (x, b) such that $t_n \to x$. Similar for $f(x-) = q$.

Theorem 2.12. $\lim_{t \to x} f(t)$ exists iff. $f(x+) = f(x-) = \lim_{t \to x} f(t)$.

Definition 2.5. (Classification of Discontinuities)
1. Discontinuity of the first kind (simple discontinuity): f is discontinuous at x, and $f(x+)$ and $f(x-)$ exist. **It can be either** $f(x+) \neq f(x-)$ **or** $f(x+) = f(x-) \neq f(x)$.
2. Discontinuity of the second kind: f is discontinuous at x, and $f(x+)$ or $f(x-)$ does not exist.

Theorem 2.13. If f is a monotonic function on (a, b), then $f(x+)$ and $f(x-)$ exist at every point of x of (a, b). Thus monotonic functions have no discontinuities of the second kind.

2.6 Monotonic Functions

Theorem 2.14. (Partial Converse to the Intermediate Value Theorem)
If f is strictly increasing on an interval I and $g(I)$ is an interval, then f is continuous on I.

Corollary 2.14.1. If f is a continuous strictly increasing function on some interval I, then f^{-1} is a continuous strictly increasing function on the interval $f(I)$.

Theorem 2.15. If f is a 1-1 continuous function on an interval I, then f is strictly increasing or strictly decreasing.

3 Sequence and Series of Functions

3.1 Basic Definitions and Theorems

Definition 3.1. We say $\{f_n\}$ converges to f pointwise on E, if

$$\forall\, x \in E,\ f(x) = \lim_{n\to\infty} f_n(x)$$

If $\sum f_n(x)$ converges for every $x \in E$, and if we define

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad (x \in E)$$

the function f is called the sum of the series $\sum f_n$.

Remark. Note that for double sequence (like $(s_{m,n})$), the limit processes ($\lim_{n\to\infty}$ and $\lim_{m\to\infty}$) may not be interchanged without affecting the result. And since to ask whether the limit of a sequence of continuous functions is continuous is to ask whether

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t)$$

This proposition only holds under certain conditions.

Also note that a convergent series of continuous functions may have a discontinuous sum.

Definition 3.2. We say $\{f_n\}$ converges uniformly on E to f if

$$\forall\, \epsilon > 0,\ \exists\, N \in \mathbb{N},\ s.t.$$

$$\forall\, n \geq N \text{ and } x \in E,\ |f_n(x) - f(x)| \leq \epsilon$$

Theorem 3.1. (Cauchy Criterion) $\{f_n\}$ converges uniformly on E to f iff.

$$\forall\, \epsilon > 0,\ \exists\, N \in \mathbb{N},\ s.t.$$

$$\forall\, n, m \geq N \text{ and } x \in E,\ |f_n(x) - f_m(x)| \leq \epsilon$$

Theorem 3.2. Suppose $\{f_n\}$ is a sequence of functions defined on E, and suppose

$$|f_n(x)| \leq M_n \quad (x \in E,\ n = 1, 2, 3, \cdots)$$

Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

3.2 Uniform Convergence and Continuity

Theorem 3.3. If $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E, then

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t)$$

Corollary 3.3.1. If $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \to f$ uniformly on E, then f is continuous on E.

3.3 Uniform Convergence and Differentiation

Theorem 3.4. Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a, b]$ and such that $|f_n(x_0)$ converges for some point x_0 on $[a, b]$. If $\{f'_n\}$ converges uniformly on $[a, b]$, then $\{f_n\}$ converges uniformly on $[a, b]$, to a function f, and

$$f'(x) = \lim_{n\to\infty} f'_n(x)$$

3.4 Uniform Convergence and Integration

Theorem 3.5. Let α be monotonically increasing on $[a, b]$. Suppose $f_n \in \mathcal{R}(\alpha)$ on $[a, b]$, for $n = 1, 2, 3, \cdots$, and suppose $f_n \to f$ uniformly on $[a, b]$. Then $f \in \mathcal{R}(\alpha)$ in $[a, b]$, and

$$\int_a^b f d\alpha = \lim_{n\to\infty}\int_a^b f_n d\alpha$$

(The existence of the limit is part of the conclusion)

Corollary 3.5.1. If $f_n \in \mathcal{R}(\alpha)$ on $[a, b]$ and if

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad (a \le x \le b)$$

the series converging uniformly on $[a, b]$, then

$$\int_a^b f d\alpha = \sum_{n=1}^{\infty}\int_a^b f_n d\alpha$$

In other words, the series may be integrated term by term.

4 Differentiation

4.1 Derivatives

Definition 4.1. Let f be defined (and real valued) on $[a,b]$. For any $x \in [a,b]$ form the quotient

$$\phi(t) = \frac{f(t)-f(x)}{t-x} \quad (a < t < b,\ t \neq x)$$

and define

$$f'(x) = \lim_{t \to x} \phi(t)$$

provided this limit exists.

Theorem 4.1. Let f be defined on $[a,b]$. If f is differentiable at a point $x \in [a,b]$, then f is continuous at x.

Theorem 4.2. Suppose f, g are defined on $[a,b]$ and are differentiable at a point $x \in [a,b]$. Then $f+g,\ fg,\ f/g$ are differentable at x, and

$$\begin{aligned} &(a) \quad (f+g)'(x) = f'(x) + g'(x); \\ &(b) \quad (fg)'(x) = f'(x)g(x) + f(x)g'(x); \\ &(c) \quad \left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}. \end{aligned}$$

In (c), we assume of course that $g(x) \neq 0$.

Theorem 4.3. (Chain Rule) If f is differentiable at x and g is differentiable at $f(x)$, then the composite function $g \circ f$ is differentiable at x and $(g \circ f)'(x) = g'(f(x))f'(x)$.

4.2 Mean Value Theorem

Definition 4.2. We say a real function f has a local maximum at a point p if there exists $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in X$ with $d(p,q) < \delta$. Local minima are defined likewise.

Theorem 4.4. Let f be defined on $[a,b]$; if f has a local maximum or local minimum at a point $x \in (a,b)$, and if $f'(x)$ exists, then $f'(x) = 0$.

Theorem 4.5. (Rolle's Theorem) Let f be a continuous function on $[a,b]$ that is differentiable on (a,b) and satisfies $f(a) = f(b)$. There exists [at least one] x in (a,b) such that f'(x)=0.

Theorem 4.6. If f and g are continuous real functions on $[a, b]$ which are differentiable in (a, b), then there is a point $x \in (a, b)$ at which

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)$$

Corollary 4.6.1. If f is continuous real functions on $[a, b]$ which are differentiable in (a, b), then there exists a point $x \in (a, b)$ at which

$$f(b) - f(a) = (b - a)f'(x)$$

Theorem 4.7. Suppose f is differentiable in (a, b).
(a) $\forall\, x \in (a, b),\ f'(x) \geq 0 \quad \Longrightarrow \quad f$ is monotonically increasing.
(b) $\forall\, x \in (a, b),\ f'(x) = 0 \quad \Longrightarrow \quad f$ is constant.
(c) $\forall\, x \in (a, b),\ f'(x) \leq 0 \quad \Longrightarrow \quad f$ is monotonically decreasing.

4.3 Continuity of Derivatives

Theorem 4.8. (Intermediate Value Theorem for Derivatives) Suppose f is a real differentiable function on $[a, b]$ and suppose $f'(a) < \lambda < f'(b)$. Then there is a point $x \in (a, b)$ such that $f'(x) = \lambda$. A similar results holds if $f'(a) > f'(b)$.

Corollary 4.8.1. If f is differentiable on $[a, b]$, then f' cannot have any simple discontinuities on $[a, b]$.

Theorem 4.9. Let f be a 1-1 continuous function on an open interval I. If f is differentiable at $x_0 \in I$ and if $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

4.4 L'Hospital's Rule

Theorem 4.10. Let s signify $a,\ a^+,\ a^-,\ \infty$ *or* $-\infty$ where $a \in \mathbb{R}$, and suppose f and g are differentiable functions for which the following limit exists:

$$\lim_{x \to s} \frac{f'(x)}{g'(x)} = L$$

If

$$\lim_{x \to s} f(x) = \lim_{x \to s} g(x) = 0$$

or if

$$\lim_{x \to s} |g(x)| = +\infty$$

then

$$\lim_{x \to s} \frac{f(x)}{g(x)} = L$$

Remark. Note that f, g must be defined and differnentiable near s and $g'(x) \neq 0$ near s.

4.5 Taylor's Theorem

Theorem 4.11. Suppose f is a real function on $[a, b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a, b]$, and define

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)^k$$

Then there exists a point x between α and β such that

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n$$