Math 104 Final Review:

Midterm 2

5. Compactness and Series: (Ross 2.13, Rudin Chapter 2)

Lecture 11

- **<u>Topology of Metric Space:</u>** Let $E \subset S$ any subset
  a. $p \in E$ is an interior point of E, $\exists \delta > 0$, such that $B\delta(p) = \{q \in S \mid d(p,q) < \delta\} \subset E$
  b. E_o=the set of interior points of E
  c. $E \subset S$ is an open subset of S, if $E = E_o$, i.e $\forall p \in E$, $\exists \delta > 0$, such that $B\delta(p) \subset E$
- **<u>Important Properties:</u>** Let (S,d) be metric space
  a. S, ø are open
  b. If $\{U\alpha\}$ is a collection of open sets, then $U\alpha G\alpha$ is open
  c. if $\{Gi\}$ open, then $\cap Gi$ open
- **<u>Complement:</u>** $E \subset S$ is a closed subset of S, if the complement $Ec = S \setminus E$ is open
  a. S, ø are closed
  b. If $\{F\alpha\}$ is a collection of closed sets, $\cap \alpha F\alpha$ is closed
  c. If $\{Fi\}$ is a collection of closed sets, then UFi is closed
- **<u>De-Morgan Law:</u>** if $A.B \subset S$ subsets, then
  a. $(AUB)c = Ac \cap Bc$
  b. $(A \cap B)c = AcUBc$
- **<u>Limit Points:</u>** Let $E \subset S$, $P \in E$ is a limit point of E if and only if $\forall \delta > 0$, $B\delta(p) = \{q \in S \mid d(p,q) < \delta\}$ intersects E non-empty, i.e. $\exists q \in E$, $d(q,p) < \delta$
- **<u>Closure:</u>** $E \subset S$, any subset, the closure of E is the intersection of all closed subsets containing E, denoted by E^-
  a. proposition: $E \cup E'$
  b. Boundary: $E^- \setminus E_o$
- **<u>Proposition 13.9</u>**: Let E be a subset of a metric space (S,d)
  a. the set E is closed if and only if $E = E^-$
  b.E is closed if and only if it contains the limit of every convergent sequence of points in E
  c.An element is in E^- if and only if it is the limit of some sequence of points in E
  d.A point is the boundary of E if and only if it belongs to the closure of both E and its complement

- **Isolated points:** If $p \in E$ and p is not a limit point of E, then p is called an isolated point
- **Perfect:** E is perfect if E is closed and every point of E is a limit point of E
- **Dense:** E is dense in S if every point of S is a limit point of E or a point of E or both.
- **Rudin2.30:** Suppose $Y \subset X$. A subset E of Y is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X
- **Compact Set:**
    a. Open cover: Let (S,d) be a metric space, $E \subset S$, $\{G\alpha\}$ is a collection of open sets. We say $\{G\alpha\}$ is an open cover of E if $E \subset \cup_\alpha G\alpha$
    b. Compact set: $K \subset S$ is a compact subset, if for any open cover of K, there exists a finite subcover, i.e if $\{G\alpha\}$ is an open cover, then $\alpha_1, \ldots, \alpha_n$ indices such that $K \subset G\alpha_1 \cup \ldots \cup \alpha_n$
    c. Sequentially Compact: $E \subset S$ is sequentially compact if any sequence in E has a convergent subsequence in E (the limit point is also in E)

- **Theorem:**
    a. for any metric space (S,d), $E \subset S$, E compact $\Leftrightarrow$ E sequentially compact
    b. (Heine-Borel theorem): consider R^n with Euclidean metric $d(x,y) = |x-y|$, $E \subset R^n$ is compact $\Leftrightarrow$ E is closed and bounded
    c. (Rudin) $K \subset Y \subset X$, the K is compact relative to Y if and only if K is compact relative to X
    d. (Rudin) Compact subsets of metric space are closed
    e. (Rudin) Closed subsets of compact sets are compact

Lecture 12

- **Series:**
    a. Infinite sum: an infinite sum of sequence (a_n) is defined as $a_1 + a_2 + a_3 + \ldots = \sum a_n$
    b. Convergence: a series converge to α if the corresponding partial sum converges to α
    c. Cauchy condition for series convergence: $\forall \epsilon > 0$, $\exists N > 0$ such that $\forall n,m > N$, $|\sum_{i=n+1} a_i| < \epsilon$
    d. Absolute Convergence: if $\sum |a_n| < \infty$, we say $\sum a_n$ converges absolutely
- **Series Convergence Tests:**
    a. Comparison Test: suppose $\sum a_n < \infty$, $a_n > 0$, and $b_n \in R < a_n$, then $\sum b_n < \infty$
       suppose $\sum a_n = \infty$, $a_n > 0$, and $b_n \in R \geq a_n$, then $\sum b_n = \infty$
    b. Ratio Test: if $\limsup |a_{n+1}/a_n| < 1$, then $\sum |a_n|$ converges
       if $\liminf |a_{n+1}/a_n| > 1$, then $\sum |a_n|$ diverges
       Otherise, no information
    c. Root Test: let $\sum a_n$ be series, $\alpha = \limsup(|a_n|)^{(1/n)}$, then $\sum a_n$:

Converges absolutely if $\alpha<1$

Diverges if $\alpha>1$

$\alpha=1$, no information

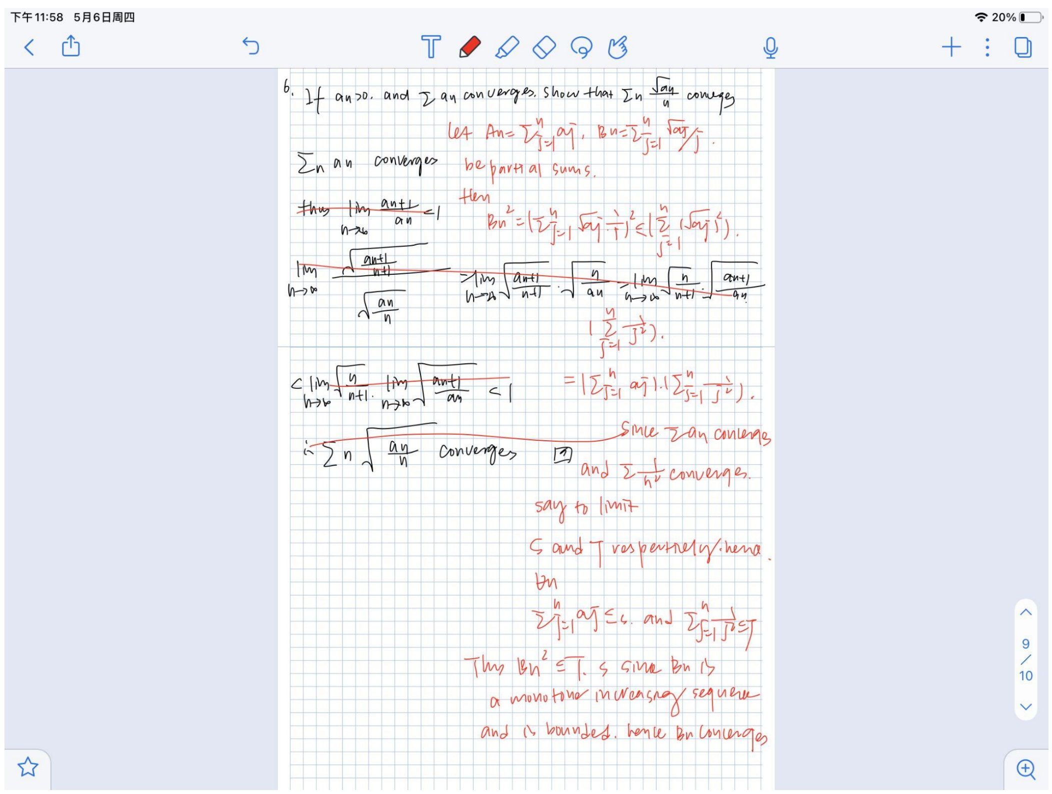
d. Alternating Series Test: let a1≥a2≥...be a monotone decreasing series, an≥0. And assuming liman=0. Then ∑(-1)^(n+1)an=a1-a2+a3-...converges. Moreover the partial sums sn=∑(-1)^(k+1)ak satisfy ∣s-sn∣≤an for all n

e. Integral Tests: if the terms are in ∑an are non-negative and f(n)=an is a decreasing function on [1,∞), then let α=lim∫f(x)dx

If $\alpha=\infty$, then the series diverge

If $\alpha<\infty$, then the series converge

Homework 6



6. Continuous functions and Compact functions

Lecture 13

- **<u>Continuous functions:</u>**

Function: A function from set A to set B is an assignment for each element $\alpha\in A$, an element $f(\alpha)\in B$

a. Injective: (one-to-one) if $\forall x,y\in A$, $x\neq y$, then $f(x)\neq f(y)$

b.surjective: if $\beta\in B$, there exists at least one element $\alpha\in A$ such that $f(\alpha)=\beta$

c. bijective: both injective and surjective

- **Limit of a function:** suppose $p \in$ set of limit points of E, we write $f(x) \to q (\in Y)$ as
- $x \to p$ or $\lim_{x \to p} f(x) = q$ if lim $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x \in E, 0 < d_X(x,p) < \delta \Rightarrow d_Y(f(x), q) < \epsilon$
- **Some Important theories from Rudin:**
    a. $\lim_{x \to p} f(x) = q$ if and only if $\lim_{n \to \infty} f(p_n) = q$ for every sequence (p_n) in E such that $p_n \neq p$, $\lim_{n \to \infty} (p_n) = p$
    b. If f has a limit at point p, then it is unique
- **Continuity of Functions:**

    a. continuity at a point: let $(X, d_X), (Y, d_Y)$ be metric spaces, $E \subset X$, $f: E \to Y$, $p \in E$, $q = f(p)$. We say f is continuous at p, if $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x \in E$ with $d_X(x,p) < \delta \Rightarrow d_Y(f(x), q) < \epsilon$

    b. If $p \in E$ is also a limit point of E, then f is continuous at $p \Leftrightarrow \lim_{x \to p} f(x) = f(p)$

    c. $(X, d_X), (Y, d_Y)$, $f: X \to Y$. Then f is continuous $\Leftrightarrow$ for every open set $V \subset Y$, $f^{-1}(V)$ is open in X.

    d. if $f: A \to B$ is a function and $E \subset A, F \subset B$. The $f(E) = F \Leftrightarrow E \subset f^{-1}(F)$

    e. Let X, Y, Z be metric spaces and $f: X \to Y$ and $g: Y \to Z$ continuous functions. We define $h: X \to Z$ by $h(x) = g(f(x))$. Then h is also continuous

    f. If $f, g: X \to R$ continuous, then $f+g, f-g, fg$ are continuous functions, and if $g(x) \neq 0$ for any $x \in X$, then f/g is also continuous.

    h. Let $f: X \to R^n$, with $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$. Then f is continuous $\Leftrightarrow f_i$ is continuous.
- **Compact Sets:**

a. Propositions: K compact $\Rightarrow$ K bounded

   K compact $\Rightarrow$ K closed

   $E \subset X$ is closed, K is compact, $E \subset K \Rightarrow E$ is compact.

b. <u>Theorems</u>: Compactness $\Leftrightarrow$ Sequential Compactness

   K compact $\Leftrightarrow$ K closed and bounded.

- **Continuous Maps and Compactness:**

  Three Definitions of Continuous Maps:

  f is continuous if and only if $\forall p \in X, \forall \epsilon > 0, \exists \delta > 0$ such that $f(B_\delta(p)) \subset B_\epsilon(f(p)$

  f is continuous if and only if $\forall V \subset Y$ open, $f^{-1}(V)$ is open

  f is continuous if and only if $\forall x_n \to x$ in X, we have $f(x_n) \to f(x)$ in Y

  Some Theories from Rudin;

a. Suppose f is a continuous map from a compact metric space X to another compact metric space Y, then $f(X) \subset Y$ is compact.
b. Suppose f is a continuous real function on a compact metric space X, and $M = \sup_{p \in X} f(p)$, $m = \inf_{p \in X} f(p)$. Then there exists point $p, q \in X$ such that $f(p) = M$ and $f(q) = m$

Homework 7

6. Uniform Continuity and Connectedness (Rudin Chap 2 and Chap 4)

Lecture 15

- **Uniform Continuous function:** f: X→Y. Suppose $\forall \epsilon>0, \exists \delta>0$ such that $\forall$ p,q$\in$X with dx(p,q)<δ, we have dY(f(p),f(q))<ϵ. We say f is a uniform continuous function.
- **Connectedness:** let X be a set. We say X is connected if $\forall$S$\subset$X and S is both open and closed, then S has to be either X or ø.
- **Propositions:**

  a. If f: X→Y is uniformly continuous and S$\subset$X subset with induced metric, then the restriction f|s:S→Y is uniformly continuous

  b.X is connected if and only if X=U⊔V and U and V are both open, then one of U,V is empty set.

  c. If f:X→Y is continuous, if E$\subset$X is connected, then f(E) is connected

  d. [0,1]$\subset$R is a connected subset.

Lecture 16

- **Continued Connectedness:** (some important conclusions from Rudin)

  a.E is connected if and only if E cannot be written as A$\cup B$ when A−∩B=ø and A∩B−=ø

  b. E$\subset$R is connected⇔∀x,y∈E,x<y, we have [x,y]$\subset$E

  c. Let f be a continuous real function on the interval [a,b]. If f(a)<f(b) and if c is a number such that f(a)<c<f(b), there exists a point x$\in$[a,b] such that f(x)=c.

- **Discontinuity:** f:$X \to Y$ is discontinuous at x$\in$X if and only if x is a limit point of X and lim$x \to pf(q)$ either does not exist or ≠f(x)

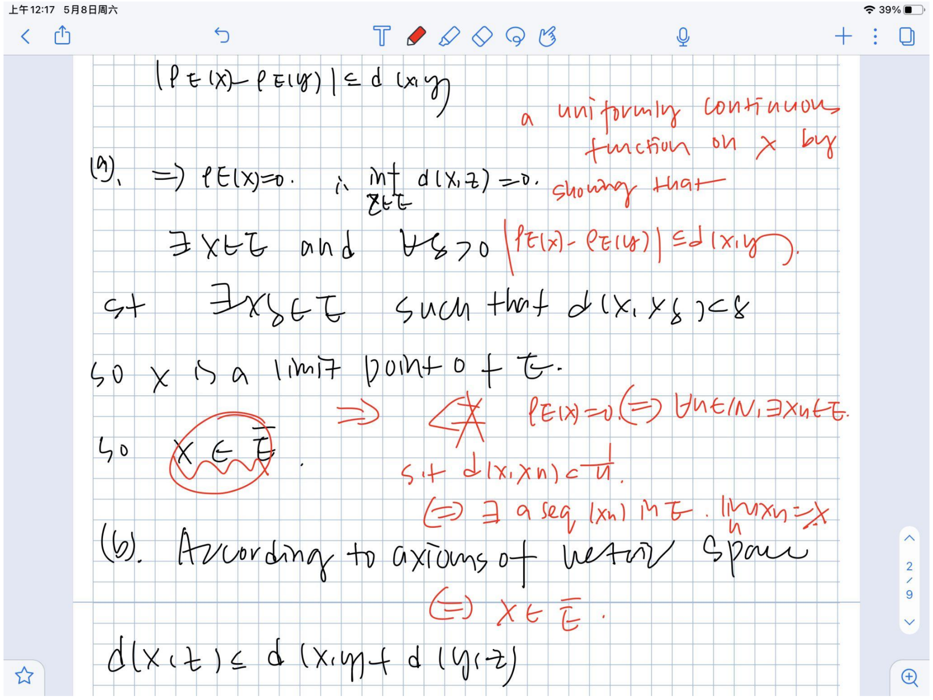
  a.Right and Left Limit: Let f:(a,b)→R $\forall$x$\in$[a,b), we say f(x+)=q if for all sequence(tn) in (x,b) that converge to x, we have limn f(tn)=q, and $\forall$x$\in$(a,b], we say f(x−)=q if for all sequence (tn) in (a,x) that converge to x, we have limn f(tn)=q

- **Discontinuity of First and Second Kind:**

  a.f:(a,b), x$\in$(a,b). Suppose f is discontinuous at x, we say f has a simple discontinuity at xo, if both f(xo+) and f(xo−) exist.

  b.We say f has a discontinuity of second kind, if it is not a simple discontinuity.

Homework 8



7. Monotonic functions and uniform convergence:

Lecture 17

- **<u>Monotonic Functions:</u>** A function f:(a,b)→R is monotone increasing if ∀x>y, we have

f(x)≥f(y). Similarly one can define monotone decreasing functions.、

- **<u>Some theorems from Rudin:</u>**

a.Suppose f:(a,b)→R is a monotone increasing function, then ∀x∈(a,b), the left limit f(x−) and the right limit f(x+) exists, satisfying sup{f(t)|t<x}=f(x−)≤f(x+)=inf{f(t)|t>x}; and given x<y in (a,b), then f(x+)≤f(y−).

b.If f is monotone, then f(x) only has discontinuity of the simple discontinuity.

c.If f is monotone, then there are at most countably many discontinuities.

- **<u>Sequence and Convergence of Functions:</u>**

  Pointwise Convergence of Sequence of Sequences: Let (xn)n be a sequence of sequences, xn∈RN, we say (xn)n converges to x∈RN pointwise if ∀i∈N, we have limn→∞xni=xi

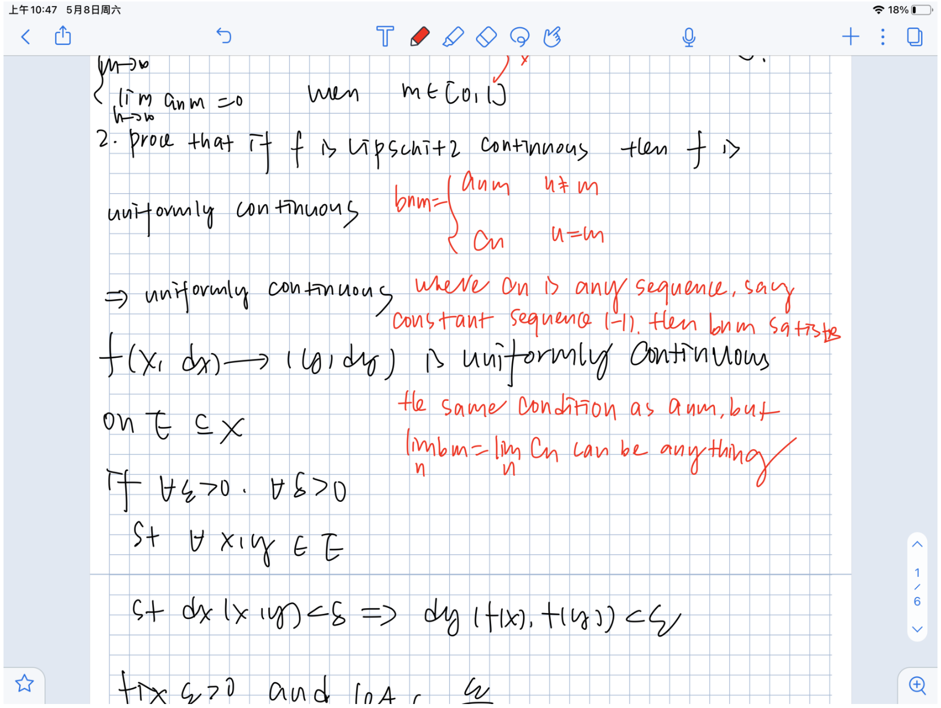
  Uniform Convergence of Sequence of Sequence: Let (xn)n be a sequence of sequences, xn∈RN, we say xn→x uniformly if ∀ϵ>0, ∃N>0 such that ∀n>N,sup{|xni−xi|:i∈N}<ϵ

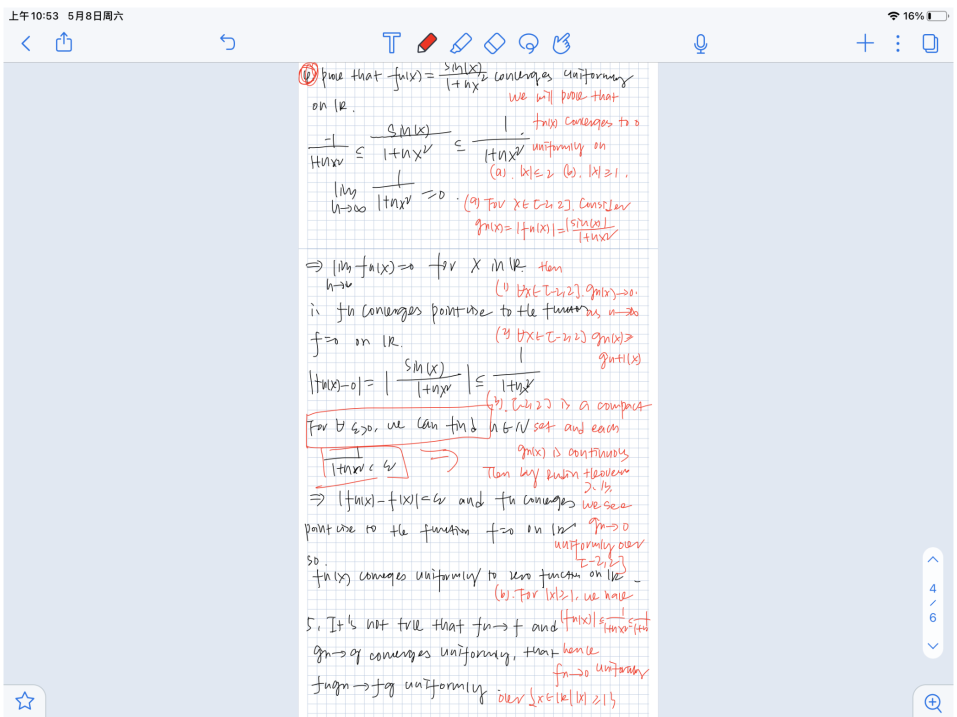
  Pointwise Convergence of Sequence of Functions: Given a sequence of functions fn∈ Map(R,R), we say fn converge to f pointwise if ∀x∈R limn→∞fn(x)=f(x)⇔n→∞|fn(x)−f(x)|=0 (examples: shrinking bumps)

Lecture 18

- **Uniform Convergence:**
  Uniform Convergence of Sequence of Functions: Given a sequence of functions $(f_n):X\to Y$, is said to converge uniformly to $f:X\to Y$, if $\forall \epsilon>0, \exists N>0$ such that $\forall n>N, \forall x\in X$, we have $|f_n(x)-f(x)|<\epsilon$
- **Some theories from Rudin**
  a.Suppose $f_n:X\to R$ satisfies that $\forall \epsilon>0, \exists N>0$ such that $\forall x\in X, |f_n(x)-f_m(x)|<\epsilon$, then f_n converges uniformly (Uniform Cauchy $\Leftrightarrow$ Uniform Convergence).
  b. (Weierstrass M-Test): Suppose $f(x)=\sum f_n(x)\,\forall x\in X$, if $\exists M_n>0$ such that $\sup_x|f_n(x)|\le M_n$ and $\sum_n M_n<\infty$, then the partial sum $F_N(x)=\sum f_n(x)$ converges to $f(x)$ uniformly.
- **Uniform Convergence and Continuity:**
  a.Suppose $f_n\to f$ uniformly on set E in a metric space. Let x be a limit point of E, and suppose that $\lim_{t\to x} f_n(t)=A_n$. Then $\{A_n\}$ converges and $\lim_{t\to x} f(t)=\lim_{n\to\infty} A_n$. $\lim_{t\to x}\lim_{n\to\infty} f_n(t)=\lim_{n\to\infty}\lim_{t\to x} f_n(t)$.
  b.If $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n\to f$ uniformly on E, then f is continuous on E
  c.Suppose K compact and
  1.$\{f_n\}$ is a sequence of continuous functions on K, 2. $\{f_n\}$ converges pointwise to a continuous function $f(x)$ on K 3. $f_n(x)\ge f_{n+1}(x)\,\forall x\in K, \forall n=1,2,\ldots n$, *then* $f_n\to f$ uniformly on K.

Homework 9





8. Derivative and Mean Value Theorem (Chapter 5 Rudin)

Lecture 21

- **Derivative:** Let f: [a,b] →R be a real valued function. Define ∀x∈[a,b], f′(x)=limt→x(f(t)−f(x)/t−x). If f′(x) exists, we say f is differentiable at this point x.
- **Proposition**:

a. If $f:[a,b]\to R$, $f:[a,b]\to R$ is differentiable at $x_o\in[a,b]$, then f is continuous at x0, i.e. $x\to x_o$ $f(x)=f(x_o)$

b. Let $f,g:[a,b]\to R$, $f,g:[a,b]\to R$. Assume f,g are differentiable at point $x_o\in[a,b]$, then

1. $\forall c\in R$, $(cf)'(x_o)=cf'(x_o)$
2. $(f+g)'(x_o)=f'(x_o)+g'(x_o)$
3. $(fg)'(x_o)=f'(x_o)g(x_o)+f(x_o)g'(x_o)$
4. if $g(x_o)\neq0$, then $(f/g)'(x_o)=f'(x_o)g(x_o)-f(x_o)g'(x_o)/(g(x_o))2$
5. (Chain Rule): Suppose $f:[a,b]\to I\subset R$ and $g:I\to R$. Suppose for some $x_o\in[a,b]$, $f(x_o)=y_o$, $y_o\in R$, $f'(x_o)$ and $g'(y_o)$exists. Then, the composition $h=g\circ f:[a,b]\to R$, $h(x)=g(f(x))$ is differentiable at x_o , $h'(x_o)=g'(y_o)f'(x_o)$.

- **<u>Mean Value Theorem:</u>**

a. Local Maximum and Minimum: Let $f:[a,b]\to R$, We say f has a local maximum at point $p\in[a,b]$, if $\exists\delta>0$ and $\forall x\in[a,b]\cap B\delta(p)$, $f(x)\leq f(p)$.

b. Let $f:[a,b]\to R$. If f has a local maximum or minimum at $p\in(a,b)$, and if f is differentiable at p, then $f'(p)=0$ (local maximum and local minimum can be taken at the endpoints!)

c. Rolle's Theorem: Suppose $f:[a,b]\to R$ is a continuous function and f is differentiable in (a,b). If $f(a)=f(b)$, then there is some $d\in(a,b)$ such that $f'(d)=0$.

Lecture 22

- **<u>Continued Mean Value Theorem:</u>** Let $f,g:[a,b]\to R$ be continuous function differentiable on (a,b). Then $\exists d\in(a,b)$ such that $[f(a)-f(b)]g'(d)=[g(a)-g(b)]f'(d)$

a. Rudin 5.10: Let $f:[a,b]\to R$ be a continuous function differentiable on (a,b). Then $\exists d\in(a,b)$ such that $[f(b)-f(a)]=[b-a]f'(d)$

b. Suppose $f:[a,b]\to R$ be continuous function,$f'(x)$ exists for all $x\in(a,b)$, and $|f'(x)|\leq M$ for some constant M.Then f is uniformly continuous.

c. Rudin 5.11: Suppose f is differentiable in (a,b), then

1. If $f'(x)\geq0$ for all $x\in(a,b)$, then f is monotonically increasing.
2. If $f'(x)=0$ for all $x\in(a,b)$, then f is constant.
3. If $f'(x)\leq0$ for all $x\in(a,b)$, then f is monotonically decreasing.

- **<u>Intermediate Value Theorem:</u>** Assume f is differentiable over [a,b] with $f'(a)<f'(b)$. Then from each $\lambda\in(f'(a),f'(b))$, there exists a $d\in(a,b)$ such that $f'(d)=\lambda'$
- **<u>L'Hospital's Rule:</u>** Suppose f and g are real and differentiable in (a,b), and $g'(x)\neq0$ for all $x\in(a,b)$, where $-\infty\leq a<b\leq\infty$. Suppose $f'(x)/g'(x)\to A$ as $x\to a$. Then if $f(x)\to0$ and $g(x)\to0$ as $x\to a$, or if $g(x)\to+\infty$ as $x\to a$, then $f(x)/g(x)\to A$ as $x\to a$.

Homework 10

1. One corollary of the intermediate value theorem for derivative is the following: if f is differentiable on $[a,b]$, then f' cannot have any simple discontinuities on $[a,b]$. Give a proof of this statement.

Prove by contradiction. Suppose there is a simple discontinuity at $x_0 \in [a,b]$. then $f'(x)$ is not equal to either the left or right limit of $f'(x)$ at $x=x_0$.

WLOG. Suppose $f'(x_0) \neq \lim_{x\to x_0^+} f'(x_0)$

let $a = f'(x_0)$. $b = \lim_{x\to x_0^+} f'(x)$. and let $\varepsilon = |a-b|/2$.

By definition of limit, there exists $\delta > 0$, such that if $x \in (x_0, x_0+\delta)$ then $|f'(x) - b| < \varepsilon$. However. this contradict with IVT for derivative applied to $[x_0, x_0+\delta/2]$

Since $f'(x_0) = a$. $|f'(x_0+\frac{\delta}{2}) - a| \geq |a-b| - |f'(x_0+\frac{\delta}{2}) - b| \geq \frac{|a-b|}{2}$

hence for $\mu = \frac{2}{3}a + \frac{1}{3}b$ between a and b. there $\exists$ $f'(v) = \mu$, $v \in (x_0, x_0+\frac{\delta}{2})$. $\Rightarrow |f'(v) - b| = |\mu - b| = \frac{2}{3}|a-b| > \frac{1}{2}|a-b|$ contradiction ▢

So f is Lipschitz continuous and $f': [a,b] \to \mathbb{R}$ bounded and cannot have simple discontinuities on $[a,b]$.

2. Let $f_n(x)$ be a sequence of differentiable

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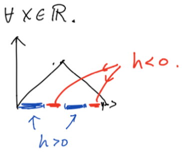
Pf: Let $h_0(x) = \begin{cases} \frac{1}{4} & x \in [n, n+\frac{1}{4}) \cup [n+\frac{1}{2}, n+\frac{3}{4}) \text{ for some } n \in \mathbb{Z} \\ -\frac{1}{4} & x \in [n+\frac{1}{4}, n+\frac{1}{2}) \cup [n+\frac{3}{4}, n+1), \text{ for some } n \in \mathbb{Z}. \end{cases}$

Then we can see that $\varphi_0(x)$ is monotonous on the interval between $x, x+h_0(x)$ $\forall x \in \mathbb{R}$.



Similarly, let $h_n(x) = 4^{-n} h_0(4^n x)$. Then we have,

If $m \leq n$, then

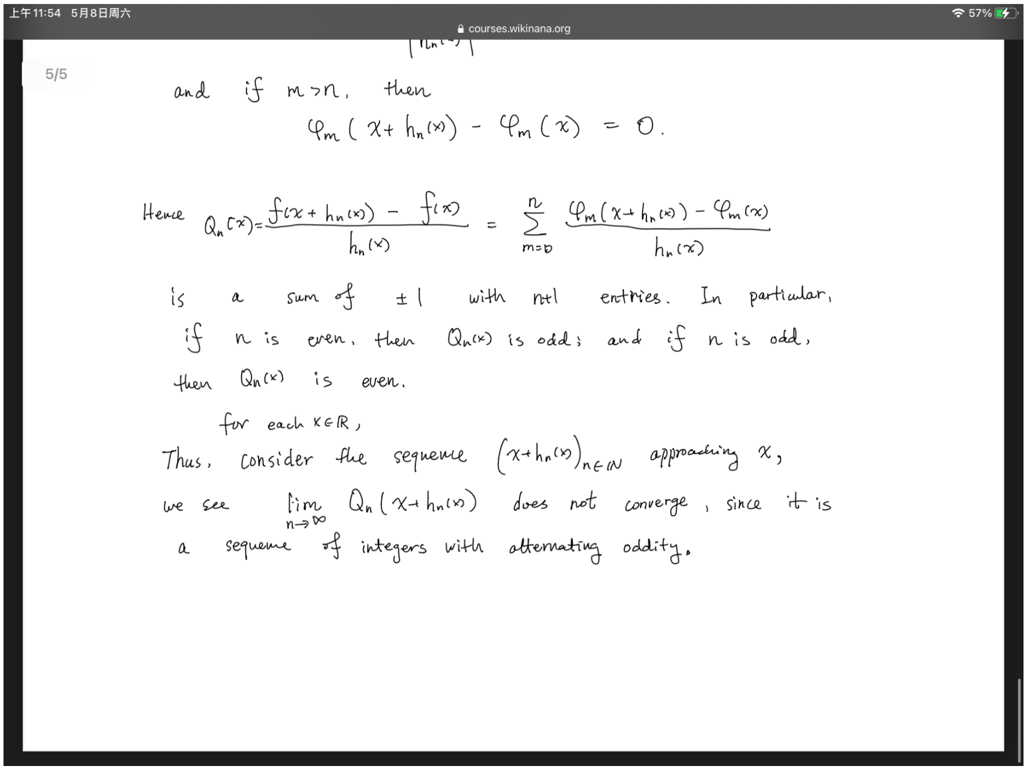
$$\frac{|\varphi_n(x+h_n(x)) - \varphi_n(x)|}{|h_n(x)|} = 1$$

and if $m > n$, then

$$\varphi_m(x+h_n(x)) - \varphi_m(x) = 0.$$

Hence $Q_n(x) = \dfrac{f(x+h_n(x)) - f(x)}{h_n(x)} = \displaystyle\sum_{m=0}^{n} \frac{\varphi_m(x+h_n(x)) - \varphi_m(x)}{h_n(x)}$

is a sum of ± 1 with $n+1$ entries. In particular,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and if $m>n$, then

$$\varphi_m(x+h_n(x)) - \varphi_m(x) = 0.$$

Hence $Q_n(x) = \frac{f(x+h_n(x)) - f(x)}{h_n(x)} = \sum_{m=0}^{n} \frac{\varphi_m(x+h_n(x)) - \varphi_m(x)}{h_n(x)}$

is a sum of ± 1 with $n+1$ entries. In particular, if n is even, then $Q_n(x)$ is odd; and if n is odd, then $Q_n(x)$ is even.

Thus, for each $x \in \mathbb{R}$, consider the sequence $(x+h_n(x))_{n\in\mathbb{N}}$ approaching x, we see $\lim_{n\to\infty} Q_n(x+h_n(x))$ does not converge, since it is a sequence of integers with alternating oddity.

9. Integrations and Differentiations:

Lecture 23

- **Higher Order Derivatives:** If f′(x)′ is differentiable at xo, we define f‘‘(xo)=(f′)′(xo).
  - a.Smooth Function: f(x) is a smooth function on (a,b)if ∀x∈(a,b), ∀k∈{1,2,…},f(k)(x) exists.
- **Taylor Theorem:** Suppose f is a real function on [a,b]. n is a positive integer, f(n−1) is continuous on [a,b], f(n)(t) exists for every t∈(a,b). Let α,β be distinct points of [a,b],, and define P(t)=∑(f(k)(α)/k!)(t−α)^k. Then there exists a point x between α and β, such that f(β)=P(β)+(f(n)(x)/n!)(β−α)^n

  *a.*Taylor Series for a Smooth Function: If f is a smooth function on (a,b), and α∈(a,b), we can form the Taylor Series: Pα(x)=∑k=(f(k)(α)/k!)(x−α)^k.

Lecture 24 (Rudin Chapter 3 and 6)

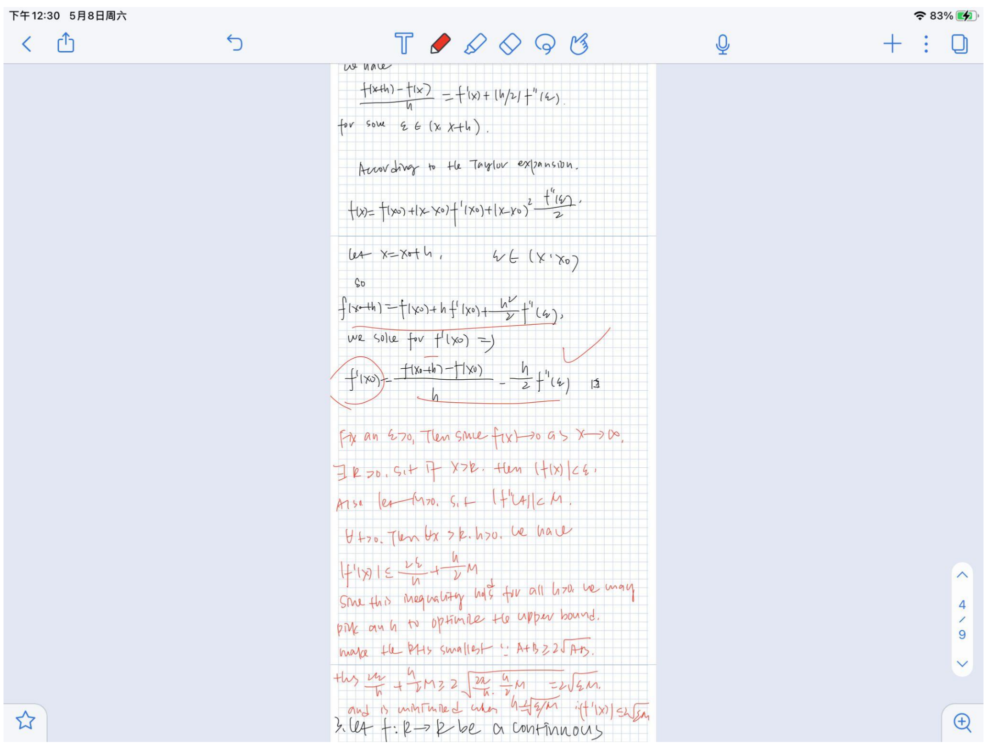
- **Taylor Series**: Let N→∞, we write Pxo(x)=∑(f(n)(xo)/n!)(x−xo)^n
  a. Consider power series ∑ncnz^n, put α=limn→∞sup|cn|^1/n. Let R=1/α, then the series is convergent if |z|<R and the series is divergent if |z|>R. Such R is called the radius of convergence.
- **Riemann Integral:**
- a.Partition: let [a,b]⊂R be a closed interval. A partition P of [a,b] is finite set of number in [a,b]: a=x0≤x1≤…≤xn= b.Define Δxi=xi−xi−1

- b.U(P,f) and L(P,f): Given f:[a,b]→R bounded, and partition p={x0≤x1≤…≤xn}, we define U(P,f)=∑ΔxiMi where Mi=sup{f(x),x∈[xi−1,xi]}; L(P,f)=∑Δximi where mi=inf{f(x),x∈[xi−1,xi]}
- c. U(f) and L(f): Define U(f)=inf$PU(P,f)$ and L(f)=sup$PL(P,f)$.Since f is bounded, hence ∃m,M∈R such that m≤f(x)≤M for all x∈[a,b], then ∀P partition of [a,b], U(P,f)≤∑ΔxiM=M(b−a),and L(P,f)≥m(b−a), and $m(b-a)\leq L(P,f) leq U(P,f)\leq M(b-a)$.
- **Riemann Integrable:** We say a function f is Riemann integrable if U(f)=L(f)

  a.If f is continuous, then f is Riemann integrable.

  b.If f is monotone, then f is Riemann integrable.

Homework 11



Lecture 25 (Continued Chapter 6)

- **Stieltjes Integrals:**
- a. Weight Function: Let α:[a.b]→R be a monotone increasing function, then α could be referred to as a weight function for Stieltjes Integral. Δαi=α(xi)−α(xi−1)
- b.Notions: we define U(P,f,α)=∑MiΔαi and L(P,f,α)=∑miΔαi

- c.*Stieltjes* Integrable: If $U(f,\alpha)=L(f,\alpha)$, we say f is integrable with respect to α and write $f\in R(\alpha)$ on [a,b].
- d.Let P and Q be 2 partitions of [a,b], then P and Q can be identified as a finite subset of [a,b]. We say Q is a refinement of P if $P\subset Q$ as subsets of [a,b].
- e.Common Refinement: Let P1 and P2 be 2 partitions of [a,b, then $P1\cup P2$ is the common refinement of P1 and P2
- **Rudin Theorem:**
- a.If P' is a refinement of P, then $L(P',f,\alpha)\leq L(P,f,\alpha)$ and $U(P',f,\alpha)\leq U(P,f,\alpha)$
- b.$L(f,\alpha)\leq U(f,\alpha)$
- *c*.$f\in R(\alpha)\Leftrightarrow\forall\epsilon>0,\exists P$ partition such that $U(P,f,\alpha)-L(P,f,\alpha)<\epsilon$
- *d.*
- 1.If c holds for P, then for any refinement Q of P, $U(Q,f,\alpha)-L(Q,f,\alpha)<\epsilon$ *and*
- 2.If c holds for P, and let $s_i,t_i\in[x_{i-1},x_i]\ \forall i=1,2,\dots,n$, then $\sum|f(s_i)-f(t_i)|\Delta\alpha_i<\epsilon$
- 3. If $f\in R(\alpha)$ and the above holds, then $\sum|f(s_i)\Delta\alpha_i-\int fd\alpha|<\epsilon$
- e. If f is continuous on [a,b], then $f\in R(\alpha)$ on [a,b]
- f. If f is monotonic on [a,b] and α is continuous, then $f\in R(\alpha)$

Lecture 26 (Continued Chapter 6 Rudin)

- **More on Integrations:**
- **a.**If f is discontinuous only at finitely many points, and α is continuous where f is discontinuous, then $f\in R(\alpha)$
- b. Let $f:[a,b]\to[m,M]$ and $\phi:[m,M]\to R$ is continuous. If f is integrable with respect to α, then $h=\phi\circ f$ is integrable with respect to α
- *1.* If $f_1,f_2\in R(\alpha)$ and $c\in R$, then $f_1+f_2, cf_1\in R(\alpha)$,and $\int f_1+f_2 d\alpha=\int f_1 d\alpha+\int f_2 d\alpha$, $\int cf_1 d\alpha=c\int f_1 d\alpha$.
- 2. If $f,g\in R(\alpha)$ and $f(x)\leq g(x),\ \forall x\in[a,b]$, then $\int fd\alpha\leq\int gd\alpha$
- 3.If $f\in R(\alpha)$ on [a,c], then $f\in R(\alpha)$ on [a,b] and on [b,c] if a<c<b, and $\int_a^c fd\alpha=\int_a^b fd\alpha+\int_b^c fd\alpha$
- 4. If $f\in R(\alpha)$ on [a,b], and $|f(x)|\leq M$ on [a,b], then $|\int_a^b fd\alpha|\leq M(\alpha(b)-\alpha(a))a$
- 5.If $f\in R(\alpha_1)$ and $f\in R(\alpha_2)$ and let c be a positive constant, then $f\in R(\alpha_1+\alpha_2)$ *and* $f\in R(c\alpha_1)$ with $\int fd(\alpha_1+\alpha_2)=\int fd\alpha_1+\int fd\alpha_2$ and $\int fd(c\alpha_1)=c\int fd\alpha_1$
- c.If $f,g\in R(\alpha)$, then $fg\in R(\alpha)$.If $f\in R(\alpha)$, then $|f|\in R(\alpha)$ and $|\int_a^b fd\alpha|\leq\int_a^b|f|d\alpha$.
- d. Unit Step function: The unit step function I is defined by $I(x)=0$ if $x\leq 0$ and $I(x)=1$ if $x>0$.
- e. If $f:[a,b]\to R f:[a,b]\to R$ and is continuous at $s\in[a,b]$ and $\alpha(x)=I(x-s)$, then $\int fd\alpha=f(s)$

- f.Suppose $c_n \geq 0$, for $n=1,2,3,\ldots \sum c_n < \infty$, $\{s_n\}$ is a sequence of distinct points in (a,b), and
  - $\alpha(x)=\sum c_n I(x-s_n)$. Let f be continuous on [a,b], then $\int f d\alpha = \sum c_n f(s_n)$