Math 104 Review

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1 Introduction

The Set of Natural Numbers, $\mathbb{N}$: The set of all positive integers (excluding 0). e.g. $\{1, 2, 3, ...\}$

The Set of Integers, $\mathbb{Z}$: The set of all integers e.g. $\{..., -2, -1, 0, 1, 2, ...\}$

The Set of Rational Numbers, $\mathbb{Q}$: The set of all rational numbers i.e. the set of all p/q where p, q $\in \mathbb{Z}$ and $q \neq 0$

The Set of Real Numbers, $\mathbb{R}$: The set of all rational and irrational real (not imaginary) numbers.

Note:

- $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
- $\emptyset$ denotes the empty set
- Rational Zeros Theorem: Let $c_0 + c_1x + ... + c_nx^n = 0$ be a polynomial equation with $n \geq 0$ and $c_0 \neq 0$, $c_n \neq 0$. Then the only rational candidates for solutions of this equation have the form a/b where a divides c_0 and b divides c_n.
- Triangle Inequality: $|a + b| \leq |a| + |b|$

Upper Bound: Let $\emptyset \neq S \subset \mathbb{R}$. We say α is a upper bound of S if $\alpha \geq \beta$ for all $\beta \in S$.

Lower Bound: Let $\emptyset \neq S \subset \mathbb{R}$. We say α is a lower bound of S if $\alpha \leq \beta$ for all $\beta \in S$.

Note:

- Upper and lower bounds may not exist.
- The infinite union of countable sets is countable.

- We define the supremum as the least upper bound of S where $sup(S) = min\{\alpha : \alpha \text{ is an upper bound of } S\}$
- We define the infimum as the greatest lower bound of S where $inf(S) = max\{\alpha : \alpha \text{ is a lower bound of } S\}$
- $\emptyset$ is bounded above and below but $sup(\emptyset) = DNE$ and $inf(\emptyset) = DNE$

The Completeness Axiom: Every set (excluding $\emptyset$) that is bounded above has a supremum. An equivalent theorem for the infimum also exists.

The Archimedean Property: If $a > 0$ and $b > 0$, then there exists $n \in \mathbb{N}$ such that $na > b$.

Density of $\mathbb{Q}$: If $a, b \in \mathbb{R}$ and $a < b$, then there is a rational number r such that $a < r < b$.

A Few Notable Examples from the Lectures, Homework, and Textbook:

1. $\sqrt{2}$ is not a rational number because, by the Rational Zeros Theorem, the only possible solutions to $x^2 - 2 = 0$ are -2, -1, 1, and 2 and none of these satisfy the equation.
2. $E = (-2, 5]$ and $F = (-2, 5)$ then $sup(E) = sup(F) = 5$ and $inf(E) = inf(F) = -2$ but $E \neq F$.
3. $E = \{q : q \in \mathbb{Q} \text{ and } q \leq \pi\}$
   $sup(E) = \pi \notin \mathbb{Q}$
   Thus the field $\mathbb{Q}$ is not complete and sets in $\mathbb{Q}$ don't need to have a rational number as an upper bound.

2 Sequences and Limits

A **sequence** is an ordered lists of real numbers $a_n \in \mathbb{R}$ that is defined for every $n \in \mathbb{N}$. A sequence is not a set.

We say a sequence (a_n) has a **limit** $\alpha \in \mathbb{R}$, if for all $\epsilon > 0$, there exists $N > 0$ such that for all $n \in \mathbb{N}$ with $n > N$, we have $|a_n - \alpha| < \epsilon$. We write $lim_{n \to \infty} a_n = \alpha$.

Squeeze Theorem: Suppose (a_n) and (b_n) are convergent sequences in $\mathbb{R}$ such that $a_n \to s$ and $b_n \to s$. If $c_n \in \mathbb{R}$ satisfies $a_n \leq c_n \leq b_n$ for all n, then $c_n \to s$.

Note:

- Sequences are useful for approximation.
- N is dependent on ϵ

- Given s_n and t_n converge to s and t, respectively:
  - For $c \in \mathbb{R}$, $lim_{n\to\infty}(cs_n) = cs$
  - $lim_{n\to\infty}(s_n + t_n) = s + t$
  - $lim_{n\to\infty}(s_n t_n) = st$
  - If $t_n \neq 0$ for all n and $t \neq 0$, then $lim_{n\to\infty}(\frac{s_n}{t_n}) = \frac{s}{t}$

A sequence (a_n) is **increasing** if $a_n \leq a_{n+1}$. A sequence (b_n) is **decreasing** if $b_n \geq b_{n+1}$.

Monotone Convergence Theorem: All bounded and monotone sequences converge.

(s_n) is a **Cauchy sequence** if for all $\epsilon > 0$, there exists $N > 0$ such that for all $n, m > N$, we have $|a_n - a_m| > 0$.

Note:

- If (s_n) is a monotone sequence, then (s_n) either converges or diverges to ∞ or$-\infty$.
- Convergent sequences are Cauchy sequences and Cauchy sequences are bounded
- A sequence (s_n) converges if and only if $limsup(s_n) = liminf(s_n)$

Ross Theorem 11.3: If the sequence (s_n) converges, then every subsequence converges to the same limit.

Define: $limsup(s_n) = lim_{N\to\infty} sup\{s_n : n > N\}$
$liminf(s_n) = lim_{N\to\infty} inf\{s_n : n > N\}$

Note:

- Let (s_n) be any sequence of nonzero real numbers:
  - $\liminf |\frac{s_{n+1}}{s_n}| \leq \liminf |s_n|^{\frac{1}{n}} \leq \limsup |s_n|^{\frac{1}{n}} \leq \limsup |\frac{s_{n+1}}{s_n}|$
  - If $\lim |\frac{s_{n+1}}{s_n}|$ exists, then $\lim |\frac{s_{n+1}}{s_n}| = \lim |s_n|^{\frac{1}{n}}$

A **metric space (S,d)** occurs when S is a set and d is a metric (function) defined for all (x,y), $x, y \in S$, that satisfies:

1. $d(x, x) = 0$ and $d(x, y) > 0$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$ $z \in S$

Bolzano-Weierstrass Theorem: Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

A Few Notable Examples from the Lectures, Homework, and Textbook:

1. Prove $\lim \frac{1}{n^2} = 0$
   Let $\epsilon > 0$ and let $N > \frac{1}{\sqrt{\epsilon}}$. Then $n > N$ implies $n > \frac{1}{\sqrt{\epsilon}}$ and hence $\epsilon > \frac{1}{n^2}$. Thus $n > N$ implies $|\frac{1}{n^2} - 0| < \epsilon$

2. Let (s_n) be a bounded sequence, show that
   $\limsup s_n = \inf\{sup_{n\geq N}(s_n) : N \in \mathbb{N}\}$.
   Let $A_N = \sup\{s_n : n \geq N\}$ and $u = \inf\{A_N : N \in \mathbb{N}\}$. Then $A_N \geq u$ for all N. And for any $\epsilon > 0$, there exists N of $\{A_n\}$ such that $u + \epsilon > A_N$.By monotonicity, $A_n > N$, we have $u+\epsilon > A_N \geq A_n \geq u$ implies $|A_n - u| < \epsilon$.

3. Let (s_n) be a sequence such that $|s_{n+1} + s_n| < 2^{-n}$ for all $n \in \mathbb{N}$. Prove that (s_n) is Cauchy.
   $|s_n - s_{n+k}| \leq |s_n - s_{n-1}| + |s_{n-1} - s_{n-2}| + ... + |s_{n-k-1} - s_{n-k}|$
   $\leq 2^{-n} + 2^{-n-1} + ... + 2^{-n-k+11} = 2^{-n}(1 + \frac{1}{2} + ... + 2^{-k+1} \leq 2^{-n} \times 2$
   Thus (s_n) is Cauchy.

3 Topology

A set E contained in a metric space S is **open** if and only if for all $x \in E$, there exists a $\delta > 0$ such that $B_\delta(x) \subset E$.

The arbitrary union of open sets is open and the intersection of finitely many open sets is open.

A set E contained in a metric space S is **closed** if every limit point of E is a point of E. A point p is a **limit point** if every neighborhood of p contains a point $q \neq p$ such that $q \in E$.

The arbitrary intersection of closed sets is closed and the union of finitely many closed sets is open.

A set $K \subset S$ is **compact** if every open cover of K contains a finite subcover.

Heine-Borel Theorem: $K \subset \mathbb{R}^n$ is compact if and only if K is closed and bounded.

A set K in a metric space S is **sequentially compact** if every sequence in K has a convergent subsequence that converges to a limit that is also in K.

A set E in a metric space X is **connected** if E is not a union of two nonempty separated sets.

Note:

- $E \subset S$ is closed if and only if E^c is open.
- $\emptyset$ and $\mathbb{R}$ are both closed and open.
- "open" and "closed" are relative terms (remember to say open in space X)
- Heine-Borel Theorem only applies to sets in $\mathbb{R}^n$
- Separated sets are disjoint but disjoint sets are not necessarily separated. e.g [0,1] and (1,2]

A Few Notable Examples from the Lectures, Homework, and Textbook:

1. Show that $K = \{1, \frac{1}{2}, ...\} \cup \{0\} \subset \mathbb{R}$ is compact.
   Let $\{G_\alpha\}$ be an open cover of K. Then there exists G_{α_0} with $0 \in G_{\alpha_0}$. There exists $\delta > 0$ such that $B_\delta(0) \subset G_{\alpha_0}$. Thus $\frac{1}{n} \in G_{\alpha_0}$ for all $\frac{1}{n} < \delta$. So there are only finitely many points in K that are not covered by G_{α_0}. Say $\frac{1}{N} < \delta$, then for all $n \leq N$, we let G_{α_n} cover the point $\frac{1}{n}$, then $\{G_{\alpha_1}, G_{\alpha_2}, ..., G_{\alpha_N}, G_{\alpha_0}\}$ is a finite subcover of K.
2. Find a subset $K \subset \mathbb{Q}$ such that K is closed and bounded in $\mathbb{Q}$ but not compact.
   $K = [0, 1] \cap \mathbb{Q}$
3. Is $\mathbb{Q}$ connected?
   Let $(-\infty, \sqrt{2}) \cap \mathbb{Q} = A$ and $(\sqrt{2}, \infty) \cap \mathbb{Q} = B$ so $A \cup B = \mathbb{Q}$. Then $(-\infty, \sqrt{2}] = \bar{A}$ and $[\sqrt{2}, \infty) = \bar{B}$ so $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$. Therefore, $\mathbb{Q}$ is not connected.

4 Series

Consider the sequences $(s_n)_{n=m}^{\infty}$ of **partial sums**:
$s_n = a_m + a_{m+1} + ... + a_n = \sum_{k=m}^{n} a_k$. Then $\sum_{n=m}^{\infty} a_n = S$ if and only if the sequence (s_n) of partial sums converges to S.

Cauchy Criterion: A series $\sum_n a_n$ satisfies the Cauchy criterion if its sequence (s_n) of partial sums is Cauchy: for each $\epsilon > 0$, there exists a number N such that $n, m > N$ implies $|s_n - s_m| < \epsilon$.

Comparison Test: Let $\sum a_n$ be a series where $a_n \geq 0$ for all n:

1. If $\sum a_n$ converges and $|b_n| \leq a_n$ for all n, then $\sum b_n$ converges.
2. If $\sum a_n = \infty$ and $b_n \geq a_n$ for all n, then $\sum b_n = \infty$.

Ratio Test: A series $\sum a_n$ of nonzero terms:

1. converges absolutely if $\limsup |\frac{a_{n+1}}{a_n}| < 1$
2. diverges if $\liminf |\frac{a_{n+1}}{a_n}| > 1$.
3. Otherwise $\liminf |\frac{a_{n+1}}{a_n}| \leq 1 \leq \limsup |\frac{a_{n+1}}{a_n}|$ and the test gives no information.

Root Test: Let $\sum a_n$ be a series and let $\alpha = \limsup |a_n|^{1/n}$. The series $\sum a_n$:

1. converges absolutely if $\alpha < 1$
2. diverges if $\alpha > 1$.
3. Otherwise $\alpha = 1$ and the test gives no information.

Alternating Series Test: If $a_{\geq} a_2 \geq ... \geq a_n \geq ... \geq 0$ and $\lim a_n = 0$, then the alternating series $\sum (-1)^{n+1} a_n$ converges. Moreover, the partial sums $s_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$ satisfy $|s - s_n| \leq a_n$ for all n.

Integral Test: $\sum_{n=1}^{\infty} \frac{1}{n^p} < \infty$ if $p > 1$.

Note:

- If $\sum a_n$ converges, then $\lim a_n = 0$.

A Few Notable Examples from the Lectures, Homework, and Textbook:

1. Show $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ when $|r| < 1$.
   $s_n = a(1 + r + ... + r^n) = a\frac{1-r^{n+1}}{1-r}$. Since $|r| < 1$, $\lim r^{n+1} = 0$ so $\lim s_n = \frac{a}{1-r}$.
2. If $a_n > 0$ and $\sum a_n$ converges, show that $\sum \frac{\sqrt{a_n}}{n}$ converges.
   Let $A_n = \sum_{j=1}^{n} a_j$ and $B_n = \sum_{j=1}^{n} \frac{\sqrt{a_j}}{j}$ be partial sums. Then
   $B_n^2 = (\sum_{j=1}^{n} \frac{\sqrt{a_j}}{j})^2 \leq (\sum_{j=1}^{n} a_j)(\sum_{j=1}^{n} \frac{1}{j^2})$. Since $\sum a_n$ and $\sum \frac{1}{n^2}$ converge, say to limit S and T respectively, then for all n, $\sum_{j=1}^{n} a_j \leq S$ and $\sum_{j=1}^{n} \frac{1}{j^2} \leq T$. Thus $B_n^2 \leq T \times S$. Thus, since B_n is a monotone increasing sequence and is bounded, B_n converges.

5 Continuity and Convergence

Let X and Y be metric spaces; suppose $E \subset X$, f maps E into Y, and p is a limit point of E. We write $f(x) \to q$ as $x \to p$, or $\mathbf{\lim_{x \to p} f(x) = q}$ if there is a point $q \in Y$ with the following property: For every $\epsilon > 0$, there exists a $\delta > 0$ such that $d_Y(f(x), q) < \epsilon$ for all points $x \in E$ for which $0 < d_X(x, p) < \delta$.

Definition One of Continuity: Suppose X and Y are metric spaces, $E \subset X$,

$p \in E$, and f maps E into Y. Then f is said to be continuous at p if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f(x), f(p)) < \epsilon$ for all points $x \in E$ for which $d_X(x, p) < \delta$.

Definition Two of Continuity: If $f : X \to Y$, then f is continuous if and only if for any limit point $p \in X$, we have $f(p) = \lim_{x \to p} f(x)$. i.e. $f(\lim_{x \to p} x) = \lim_{x \to p} f(x)$.

Definition Three of Continuity: A mapping f of a metric space X into a metric space Y is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in Y. There is an equivalent definition for closed sets.

Rudin Theorem 4.14 and 4.19: Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then f(x) is compact and f is uniformly continuous on X.

Let f be a mapping of a metric space X into a metric space Y. f is **uniformly continuous** on X if for every $\epsilon > 0$ there exists $\delta > 0$ such that $d_Y(f(p), f(q)) < \epsilon$ for all p and q in X for which $d_X(p, q) < \delta$.

Rudin Theorem 4.22: If f is a continuous mapping of a metric space X into a metric space Y, and if E is a connected subset of X, then f(E) is connected.

Intermediate Value Theorem: Let f be a continuous real function on the interval [a, b]. If $f(a) < f(b)$ and if c is a number such that $f(a) < c < f(b)$, then there exists a point $x \in (a, b)$ such that $f(x) = c$.

Let f be defined on (a, b) and f has a **simple discontinuity** at x then either:

1. $f(x+) \neq f(x-)$
2. $f(x+) = f(x-) \neq f(x)$

Suppose $\{f_n\}$ is a sequence of functions on a set E, and suppose that the sequence of numbers $\{f_n(x)\}$ **converges pointwise** for every $x \in E$. We can then define a function f by $f(x) = \lim_{n \to \infty} f_n(x)$.

We say that a sequence of functions $\{f_n\}$ **converges uniformly** on E to a function f if for every $\epsilon > 0$ there is an integer N such that $n \geq N$ implies $|f_n(x) - f(x)| \leq \epsilon$ for all $x \in E$.

Cauchy Criterion for Uniform Convergence: The sequence of functions $\{f_n\}$, defined on E, converges uniformly on E if and only if for every $\epsilon > 0$ there exists an integer N such that $m \geq N$, $n \geq N$, $x \in E$ implies $|f_n(x) - f_m(x)| \leq \epsilon$.

Weierstrass M-Test: Suppose $\{f_n\}$ is a sequence of functions defined on E, and suppose $|f_n(x)| \leq M_n$ $(x \in E, n = 1, 2, ...)$. Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

Rudin Theorem 7.12: If $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \to f$ uniformly on E, then f is continuous on E.

Rudin Theorem 7.13: Suppose K is compact, and

1. $\{f_n\}$ is a sequence of continuous functions on K,
2. $\{f_n\}$ converges pointwise to a continuous function f on K,
3. $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$, n= 1, 2,...

Then $f_n \to f$ uniformly on K.

Note:

- Let f and g be continuous functions on X, then f+g,fg, and f/g are continuous on X.
- The preimage of a compact set may not be compact.
- If $\sum f_n(x)$ converges for every $x \in e$, and if we define $f(x) = \sum_{n=1}^{infty} f_n(x)$, the function f is called the sum of the series.
- If pointwise convergence, N depends on ϵ and x. If uniform convergence, N depends only on ϵ.

A Few Examples from the Lectures, Homework, and Textbook:

1. Is $K = (0, 1]$ compact in $X \subset \mathbb{R}$?
   We know that K is bounded in X. Also, K is closed in X because $(0, 1] = X \cap [0, 1]$ and [0, 1] is closed in $\mathbb{R}$. But Heine-Borel Theorem does not apply here because $X \neq \mathbb{R}^n$. K cam be covered by $\{B^X_{\frac{1}{2n}} \frac{1}{n}\}$ for $n \in \mathbb{N}$ but this does not have a finite subcover so K is not compact which verifies that since K is not closed in $\mathbb{R}$, K is not compact.
2. If $K \subset \mathbb{R}^n$ is compact and $C \subset \mathbb{R}^n$ is closed, prove that K+C is closed.
   We only need to show that if $p_n \in K + C$ converges to $p \in \mathbb{R}^n$, then $p \in K + C$. Define each $p_n = x_n + y_n$, $x_n \in K$ and $y_n \in F$. Then we may assume that $x_n \to x \in K$. Then $y_n = p_n - x_n$ with $p_n \to p$ and $x_n \to x$. So y_n converges to $p-x$. Since C is closed, $y \in C$. Thus $p = x+y \in K+C$.
3. Prove that if $f : X \to \mathbb{R}$ is Lipschitz continuous, then f is uniformly continuous.
   By Lipschitz continuity of f, we know that there exists a $K > 0$ such that $|f(x) - f(y)| \leq K \times d(x, y)$. Hence for all $\epsilon > 0$, we may choose $\delta = \frac{\epsilon}{K}$ so that for any $x, y \in X$ with $|x - y| < \delta$, we have
   $|f(x) - f(y)| \leq K \times d(x, y) < K\delta = \epsilon$.

4. Let $f_n, g_n : X \to \mathbb{R}$ be sequences of continuous functions. Suppose $f_n \to f$ and $g_n \to g$ uniformly. Is it true that $f_n g_n \to fg$ uniformly?
   No, we can write $f_n(x) = f(x) + \alpha_n(x)$, $g_n = g(x) + \beta_n(x)$. Then $\alpha_n \to 0$ and $\beta_n \to 0$ uniformly. Then $f_n g_n = (f(x) + \alpha_n(x))(g(x) + \beta_n(x)) = fg + f\beta_n + \alpha_n g + \alpha_n \beta_n$ but $f\beta_n$ and $g\alpha_n$ may not converge to 0 uniformly. e.g. $f_n(x) = x$, $g_n(x) = \frac{1}{n}$ so $f(x) = x$ and $g(x) = 0$. However $f_n g = \frac{x}{n}$ does not converge to $fg = 0$ uniformly.
   Note that this statement is true if X is a compact set.

6 Differentiation

Let f be defined (and real-valued) on [a, b]. For any $x \in [a, b]$, define
$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$ $(a < t < b, t \neq x)$.

Generalized Mean Value Theorem: Let $f, g : [a, b] \to \mathbb{R}$ be continuous on [a, b] and differentiable on (a, b) then there exists $c \in (a, b)$ such that $(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$. If $g'(c) \neq 0$, then $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$.

Intermediate Value Theorem for f'(x): Suppose f is areal differentiable function on [a, b] and suppose $f'(a) < \lambda < f'(b)$. Then there is a point $x \in (a, b)$ such that $f'(x) = \lambda$.

L'Hospital's Rule: Suppose f and g are real and differentiable in (a, b), and $g'(x) \neq 0$ for all $x \in (a, b)$, where $-\infty \leq a < b \leq \infty$. Suppose $\frac{f'(x)}{g'(x)} \to A$ as $x \to a$. If $f(x) \to \infty$ and $g(x) \to 0$ as $x \to a$, or if $g(x) \to \infty$ as $x \to a$, $\frac{f(x)}{g(x)} \to A$ as $x \to a$.

Taylor's Theorem: Suppose f is a real function on [a, b], n is a positive integer, $f^{(n-1)}$ is continuous on [a,b], $f^{(n)}(t)$ exists for every $t \in (a, b)$. Let α, β be distinct points of [a, b], and define $P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)(t-\alpha)^k}{k!}$. Then there exists a point x between α and β such that $f(\beta) = P(\beta) + \frac{f^{(n)}(x)(\beta-\alpha)^n}{n!}$.

Taylor Series of f at x_0 as $N \to \inf$ is $P_{x_0}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)(x-x_0)^n}{n!}$.

A **power series** is a series of the form $\sum_{n=0}^{\infty} c_n(x - x_0)^n$ with **radius of convergence** $R = sup\{r \geq 0$ such that if $|x - x_0| \leq r$, the series converges$\}$.

Note:

- Let $f : [a, b] \to \mathbb{R}$, if f has a local max (or local min) at a point $x \in (a, b)$ and if $f'(x)$ exists, then $f'(x) = 0$.
- If $[a, b] \in \mathbb{R}$ is compact, $f([a, b])$ is compact.

- If f is differentiable on [a, b], then f' cannot have any simple discontinuities on [a, b].
- Taylor's Theorem with n=1 gives the Mean Value Theorem.
- To estimate the error of the constant approximation, $f(x) - P_{\alpha,0}(x) = (x - \alpha) \times f'(c)$ for c between x and a.
- To estimate the error of the linear approximation, $f(x) - P_{\alpha,1}(x) = (x - \alpha)^2 \times \frac{f''(c)}{2}$ for c between x and a.
- A smooth function f means $f^{(n)}(x)$ exists for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$.
- The Taylor Series may not converge for $x \in \mathbb{R}$ and even if the series converges for $x \in \mathbb{R}$, it may not equal to f(x).
- If $f'(a) = f'(b) = 0$, it is not possible to have a $c \in (a, b)$ such that $f'(c) = 0$.

Rudin Theorem 7.17: Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a, b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a, b]$. If $\{f'_n\}$ converges uniformly on $[a, b]$, then $\{f_n\}$ converges uniformly on $[a, b]$, to a function f, and $f'(x) = \lim_{n\to\infty} f'_n(x)$ $(a \leq x \leq b)$.

Note:

- There exists a real continuous function on the real line which is nowhere differentiable.

A Few Notable Examples from the Lectures, Homework, and Textbook:

1. Let $f : [a, b] \to \mathbb{R}$ be differentiable. Prove that $f'(x)$ cannot have any simple discontinuities.
   Proof by contradiction. Suppose there is a simple discontinuity at $x_0 \in (a, b)$, then $f'(x_0)$ is not equal to either the left or right limit of $f'(x)$ at $x = x_0$. Without loss of generality, suppose $f'(x_0) \neq \lim_{x\to x_0^+} f'(x_0)$. Let $a = f'(x_0)$, $b = \lim_{x\to x_0^+} f'(x)$, and let $\epsilon = \frac{|a-b|}{2}$. By the definition of a limit, there exists $\delta > 0$, such that if $x \in (x_0, x_0+\delta)$, then $|f'(x)-b| < \epsilon$. However, this contradicts the Intermediate Value Theorem for derivatives when applied to the interval $[x_0, x_0+\frac{\delta}{2}]$. Since $f'(x_0) = a$, $|f'(x_0+\frac{\delta}{2})-a| \geq |a - b| - |f'(x_0 + \frac{\delta}{2}) - b| \geq \frac{|a-b|}{2}$. Hence for $\mu = \frac{2a}{3} + \frac{b}{3}$ between a and b, there exists $f'(\gamma) = \mu$ with $\gamma \in (x_0, x_0 + \frac{\delta}{2})$. This means that $|f'(\gamma) - b| = |\mu - b| = \frac{2}{3}|a - b| > \frac{|a-b|}{2} = \epsilon$. This is a contradiction.
2. If a sequence of differentiable functions converges uniformly, does it mean that $f(x)$ is differentiable.
   No. Consider $f(x) = max\{0, x\}$, $x \in \mathbb{R}$ and $f_n(x) = \frac{1}{n} log(1 + e^{nx})$. Then each $f_n(x)$ is smooth and $f_n(x)$ converges uniformly to f.

3. If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f'(x)$ exists for all $x \neq 0$. If we also know that $\lim_{x\to 0} f'(x) = 5$. Show that $f'(0) = 5$.
   We need to show that $\lim_{h\to 0} \frac{f(h)-f(0)}{h} = 5$.
   Apply the Mean Value Theorem to the interval $[0, h]$. Assume that $h > 0$ (the $h < 0$ case is similar), then there exists $\gamma \in (0, h)$ such that $\frac{f(h)-f(0)}{h} = f'\gamma$. Since $\lim_{x\to 0} f'(x) = 5$, hence for all $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |x| < \delta$, then $|f'(x) - 5| < \epsilon$. Thus, if $0 < h < \delta$, we have $|\frac{f(h)-f(0)}{h} - 5| = |f'(\gamma) - 5| < \epsilon$. Hence $\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = 5$. Similarly (for $h < 0$), $\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = 5$. Thus $f'(0) = 5$.

7 Integrability

Let [a, b] be a given interval. By a **partition** P of [a,b] we mean a finite set of points $x_0, x_1, ..., x_n$, where $a = x_0 \leq x_1 \leq ... \leq x_n = b$. Now suppose f is a bounded real function defined on [a, b]. Corresponding to each partition P of [a, b] we put $M_i = \sup f(x)$ and $m_i = \inf f(x)$ for $(x_{i-1} \leq x \leq x_i)$ and $U(P, f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$, $L(P, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$.

If $\inf U(P, f) = \sup L(P, f)$, then f is **Riemann-integrable** on [a, b].

If $U(P, \alpha) = L(P, \alpha)$, we say f is **Riemann-Stieltjes integrable** with respect to α. We write this as $f \in \mathscr{R}(\alpha)$ and $\int_a^b f(x)d\alpha(x)$.

Cauchy Condition: $f \in \mathscr{R}(\alpha)$ on $[a, b]$ if and only if for every $\epsilon > 0$ there exists a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$.

Rudin Theorem 6.10: Suppose f is bounded on [a, b], f has only finitely many points of discontinuity on [a, b], and α is continuous at every point at which f is discontinuous. Then $f \in \mathscr{R}(\alpha)$.

Change of Variable Theorem: Suppose ϕ is a strictly increasing continuous function that maps an interval [A, B] onto [a,b]. Suppose α is monotonically increasing on [a, b] and $f \in \mathscr{R}(\alpha)$ on [a, b]. Define β and g on [A, B] by $\beta(y) = \alpha(\phi(y))$, $g(y) = f(\phi(y))$. Then $g \in \mathscr{R}(\beta)$ and $\int_a^b g d\beta = \int_a^b f d\alpha$.

The Fundamental Theorem of Calculus: If $f \in \mathscr{R}$ on $[a, b]$ and if there is a differentiable function F on $[a, b]$ such that $F' = f$, then
$\int_a^b f(x)dx = F(b) - F(a)$.

Integration by Parts: Suppose F and G are differentiable functions on $[a, b]$, $F' = f \in \mathscr{R}$ and $G' = g \in \mathscr{R}$. Then
$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx$.

Rudin Theorem 7.16: Let α be monotonically increasing on $[a, b]$. Suppose $f_n \in \mathscr{R}(\alpha)$ on $[a,b]$, for $n = 1,2,...$, and suppose $f_n \to f$ uniformly on $[a,b]$. Then $f \in \mathscr{R}(\alpha)$ on $[a,b]$, and $\int_a^b f d\alpha = \lim_{n\to\infty} \int_a^b f_n d\alpha$.

Rudin Theorem 7.16 Corollary: If $f_n \in \mathscr{R}(\alpha)$ on $[a,b]$ and if $f(x) = \sum_{n=1}^{\infty} f_n(x)$ $(a \leq x \leq b)$, the series converging uniformly on $[a,b]$, then $\int_a^b f d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n d\alpha$. i.e. The series may be integrated term by term.

Note:

- If f is uniformly continuous, then $f \in \mathscr{R}(\alpha)$.
- If f is monotonic and α is continuous then $f \in \mathscr{R}(\alpha)$.
- The integration operation $\int f d\alpha$ is linear in both f and α.
- The Riemann-Stieltjes integral assigns weight to an interval $I = [c,d]$, $\alpha(I) = \alpha(d) - \alpha(c)$. Integral $\int_a^b f d\alpha$ is approximated by a "weighted" sum, $\sum_i f(I_i)\alpha(I_i)$
- It is possible to have a function F such that $F'(x)$ exists for all $x \in [a,b]$ and F' is bounded but $F'(x)$ is not integrable.

A Few Notable Examples from the Lectures, Homework, and Textbook:

1. Let $f(x)$ be a function on $[0,1]$, with $f(x) = \{ \begin{array}{cc} 0 & x=0 \\ sin(\frac{1}{x}) & x>0 \end{array}$ and let α be given by $\alpha(x) = \{ \begin{array}{cc} 0 & x=0 \\ \sum_{\frac{1}{n}<x} 2^{-n} & x>0 \end{array}$.
   Show that $\int_0^1 f(x)d\alpha(x)$ exists.
   Since $\sum_{n=1}^{\infty} 2^{-n} < \infty$, hence as $N \to \infty$, $\sum_{n=N}^{\infty} 2^{-n} \to 0$. Thus, as $x \to 0$, $\frac{1}{x} \to \infty$ so $\alpha(x) = \sum_{n>\frac{1}{x}} 2^{-n} \to 0$. Thus $\alpha(x)$ is continuous at $x = 0$. Since $f(x)$ is a bounded real function with discontinuity only at $x = 0$ and α is continuous at $x = 0$. Thus, by Rudin Theorem 6.10, $f \in \mathscr{R}(\alpha)$.

2. Suppose f is a continuous non-negative function on $[a,b]$. Show that $\int_a^b f(x)dx = 0$ implies $f(x) = 0$ for all $x \in [a,b]$.
   It suffices to show that there exists $x_0 \in [a,b]$, that $f(x_0) > 0$. Let $\epsilon = \frac{f(x_0)}{2}$ then there exists $\delta > 0$ such that for all $x \in [a,b]$, $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$. Thus $f(x) > f(x_0) - \epsilon = \frac{f(x_0)}{2}$, for all $x \in B_\delta(x_0) \cap [a,b]$. Let $[c,d] \subset B_\delta(x_0) \cap [a,b]$ for $d > c$. Then $\int_a^b f(x)dx \geq \int_c^d f(x)dx \geq \int_c^d \epsilon dx = \epsilon(d-c) > 0$. This contradicts with $\int_a^b f dx = 0$.

8 Questions

1. **Why is $\mathbb{N}$ unbounded in $\mathbb{R}$?**
   Answer: Assume $\mathbb{N}$ is bounded. Then $\alpha = sup(\mathbb{N})$ implies that $n \leq \alpha$ for

all $n \in (N)$. We know that $n = 1 \in \mathbb{N}$ which means $n+1 \leq \alpha$ so $n \leq \alpha - 1$ for all n. Thus $\alpha - 1$ is a least upper bound of $\mathbb{N}$ which is a contradiction.

2. **Prove that $|a - b| \geq ||a| - |b||$.**
Answer: $|a| = |a-b+b| \leq |a-b|+|b|$ and $|b| = |b-a+a| \leq |b-a|+|a|$ by the Triangle Inequality. Then $|a|-|b| \leq |a-b|$ and $|b|-|a| \leq |b-a| = |a-b|$ which implies that $|a| - |b| \geq -|a - b|$ so $-|a - b| \leq |a| + |b| \leq |a - b|$. Thus $|a - b| \geq ||a| - |b||$.

3. **Does there exist a sequence that has an infinite number of x's with $x \in \mathbb{R}$ that converges to a limit other than x?**
Answer: Let $x_n \to L$ for $L \neq x$. Then let $\epsilon = \frac{|x-L|}{2} > 0$ so x is not in the ϵ-neighborhood. However, this is a contradiction because the ϵ-neighborhood contains all but finitely many values of (x_n) but there are infinitely many x's.

4. **Show that $(\frac{-\pi}{2}, \frac{\pi}{2})$ and $\mathbb{R}$ have the same cardinality.**
Answer: Let $f(x) = arctan(x)$, then $f(0) = 0$ and $f(1) = \frac{\pi}{4}$. Then $f(x)$ is a continuous and strictly increasing function. Also $f(x)$ is injective and onto and maps $(\frac{\pi}{2}, \frac{-\pi}{2})$ to $\mathbb{R}$.

5. **If the sequence (s_n) is bounded and the sequence (t_n) converges to $t \neq 0$, does $lim_{n\to\infty}$ necessarily exist?**
Answer: No, let $(s_n) = (-1)^n$ which is bounded by -1 and 1 and let $(t_n) = \frac{1}{n} + 7$ which converges to 7 as $n \to \infty$. Then, $(s_n t_n) = \frac{(-1)^n}{n} + 7(-1)^n$. Thus $(s_n t_n)$ diverges.

6. **Give an example of a set that has both a supremum and a maximum, a supremum and no maximum, and no supremum and a maximum.**
Answer: Both supremum and maximum exist: [0,1]. Only supremum exists: [0,1]. Only maximum exists: such a set does not exist.

7. **What is the difference between limsup and sup?**
Answer: Supremum is the least upper bound of a set and denotes a value while limsup is an operation that takes the limit of the "tail" of a sequence.

8. **Construct a sequence (s_n) with $\limsup s_n = 1$ and $\liminf s_n = -\infty$.**
Answer: Let $s_n = \{ \begin{array}{ll} 1+\frac{1}{n} & \text{n is even} \\ -n^2 & \text{n is odd} \end{array}$

9. **Are Cauchy sequences necessarily monotone?**
Answer: No, $s_n = \frac{(-1)^n}{n}$ is not monotone but s_n converges to 0 which implies that (s_n) is Cauchy.

10. **If two sequences (s_n) and (t_n) are Cauchy and $t_n \neq 0$ for all n. Is $\frac{s_n}{t_n}$ Cauchy?**
Answer: No. $t > 0$ but (t_n) could converge to 0.

e.g. Let $t_n = 1/n$ and $s_n = \frac{1}{n} + 1$ so $s_n \to 1$ as $n \to \infty$. Then $\frac{s_n}{t_n} = 1 + n$ which diverges to ∞.

11. **(From UC Berkeley Past Exam Archive from Math Department) Show that $H_n = \sum_{k=1}^{n} \frac{1}{k}$ is not Cauchy.**
Answer: $H_{2n} = \sum_{k=1}^{2n} \frac{1}{k}$. Then $H_{2n} - H_n = \sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} = \sum_{k=n+1}^{2n} \frac{1}{k} = \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n}$ which has n terms. So $\frac{1}{2n} < \frac{1}{n+k}$ for $k = 1, ..., n-1$ which implies that $H_{2n} - H_n > \frac{n}{2n} = 1/2$.
Thus for $n, m > N$, $|H_n - H_m| < \epsilon$. When $m = 2n$, this inequality fails for $\epsilon < 2$, therefore H_n is not Cauchy.

12. **What is a countable set?**
Answer: A set E is countable if it has a finite number of elements or if every element in E maps to a value in $\mathbb{N}$.

13. **Is $\mathbb{Z}$ countable?**
Answer: Yes. $\mathbb{Z} = \mathbb{N} \cup -\mathbb{N} \cup \{0\}$. We know that $\mathbb{N}$ is countable so $-\mathbb{N}$ is countable and $\{0\}$ is countable. Lastly, the union of countable sets is countable. We could also create a one-to-one mapping between $\mathbb{N}$ and $\mathbb{Z}$ with $1 \to 0$, $2 \to 1$, $3 \to -1$, $4 \to 2$, $5 \to -2$ and so on.

14. **What is a field?**
Answer: A system that has more than one element and satisfies the nine associative, distributive, and commutative laws.

15. **Does every unbounded sequence have a divergent subsequence?**
Answer: Yes. If $(|a_n|)$ is unbounded, then for every $M \in \mathbb{R}$, we have $|a_n| > M$. Let $M = 1$ and choose $|a_{n_1}| > 1$. Then let $M = 2$ and choose $|a_{n_2} > 2|$ and repeat this procedure to get $|a_{n_k}| \to \infty$. Thus we have found a subsequence of unbounded $|a_n|$ that diverges to ∞.

16. **If a set E is not open, does that imply that its complement is not closed?**
Answer: Yes because E^c closed would imply $(E^c)^c = E$ is open.

17. **Are $\mathbb{R}$ and $\emptyset$ the only sets in $\mathbb{R}$ that are both open and closed?**
Answer: Suppose $E \subset \mathbb{R}$ is both open and closed and $E \neq \mathbb{R}$ and $E \neq \emptyset$. Then let $x \in E$ and $y \notin E$ and without loss of generality, assume $x < y$. Let $F = [x, y]$, then $E \cap F$ is closed. Let $z = supE \cap F$ and $z \in E \cap F$, also $z < y$. But since E is open, there exists $\epsilon > 0$ such that $B_\epsilon(z)$ is in E. But then $z < z + \epsilon \in E$ which is a contradiction.

18. **Show that the intersection of infinitely many open sets is not open.**
Answer: Let $x \in \bigcap_{n=1}^{\infty} E_n$ and let $E_n = (\frac{-1}{n}, \frac{1}{n})$ for all $n \in N$. Let $\epsilon > 0$. Then find $N \in \mathbf{N}$ such that $\frac{1}{N} < \epsilon$. Then $x \in (\frac{-1}{N}, \frac{1}{N})$ implies that $|x| < \epsilon$. Because ϵ is arbitrary, $x = 0$. Hence, $\bigcap_{n=1}^{\infty} = \{0\}$ which is not open.

19. **What is an open cover?**
Answer: Let A be a nonempty subset of a metric space S. A sequence of open sets (G_n) is an open covering of A if $A \subseteq \bigcup G_n$.

20. **What is a finite subcover?**
Answer: Let (G_n) be an open cover of A. If (G_{n_k}) is a subset of (G_n) and is also an open cover of A such that $A \subseteq \bigcup(G_{n_k})$, then (G_{n_k}) is a subcover of A. If (G_{n_k}) has finitely many terms, then it is a finite subcover of A.

21. **How are compactness and sequential compactness related?**
Answer: They are equivalent! A subset of a metric space is compact if and only if it is sequentially compact.

22. **(From UC Berkeley Past Exam Archive from Math Department) If A and B are connected in $\mathbb{R}$ and $A \cap B \neq \emptyset$, prove that $A \cap B$ is connected.**
Answer: Assume that $A \cup B$ is not connected. Then there exist two nonempty sets $E, F \subset A \cup B$ such that $E \cap F = \emptyset$. So $A \cup B = E \cup F$. Let $x \in A \cap B$ and without loss of generality, assume $x \in A$. $F \subset A \cup B$ is nonempty so assume $F \cap A$ is nonempty. Define $E' = E \cap A$ and $F' = F \cap A$ which are nonempty. Then $E' \cap F' = (E \cap A) \cap (F \cap A) = (E \cap F) \cap A = \emptyset \cap A = \emptyset$ and $E' \cup F' = (E \cap A) \cup (F \cap A) = (E \cap F) \cup A = \emptyset \cup A = A$. Thus E' and F' separate A which is a contradiction.

23. **What is Lipschitz continuity? How does it differ from pointwise or uniform continuity?**
Answer: A function f is Lipschitz continuous if and only if $f : (a, b) \to \mathbb{R}$ is differentiable and its derivative $f' : (a, b) \to \mathbb{R}$ is bounded. Thus, Lipschitz continuity implies uniform and pointwise continuity.

24. **Show that $f(x) = x^2$ is uniform continuous on (0,1) but not on $\mathbb{R}$.**
Answer: Let $\epsilon > 0$ and $x, y \in (0, 1)$, then there exists $\delta > 0$ such that $|x - y| < \delta$ and we know that $|x + y| \leq |x| + |y| \leq 2$ by the Triangle Inequality. Then $|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y| < 2 \times \delta$For $\delta = \frac{\epsilon}{2}$, $|x^2 - y^2| < \epsilon$ for all x, y.
Assume f is uniform continuous on $\mathbb{R}$.Then by definition, for $\epsilon = 1$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|x^2 - y^2| < \epsilon = 1$. Then let $x = n$ and $y = n + \frac{\delta}{2}$ so $|n^2 - (n^2 + \delta n + \frac{\delta^2}{4})| = | - \delta n - \frac{\delta^2}{4}| = \delta n + \frac{\delta^2}{4}$ for all $n \in \mathbb{N}$. But $\delta n + \frac{\delta^2}{4} \nleq 1$ for all $n \in \mathbb{N}$ so f is not uniform continuous on $\mathbb{R}$.

25. **Let be $A \subset \mathbb{R}$ bounded and $f : A \to \mathbb{R}$ be uniformly continuous. Is f bounded on A?**
Answer: Suppose that f is unbounded. For each $n \in \mathbb{R}$, there exists $x_n \in A$ where $|f(x_n)| > n$. Now, $|f(x_n)| \to \infty$ but (x_n) is bounded and by Bolzano-Weierstrass Theorem, (x_n) has a convergent subsequence

(x_{n_k}). (x_{n_k}) Cauchy which implies that $(f(x_{n_k}))$ is Cauchy and converges and therefore bounded. So $|f(x_{n_k})| \leq M$ for all k. But if $k > M$, then $|f_{n_k}| > n_k \geq k > M$ which is a contradiction.

26. **What is a Riemann-Stieltjes integral?**
Answer: A Riemann-Stieltjes integral is a generalization of a Riemann integral where instead of integrating in terms of some variable x, we integrate in terms of a function α.

27. **What is a Lebesgue integral?**
Answer: To find the area under a curve, a Riemann integral subdivides the x-axis into vertical rectangles whereas a Lebesgue integral subdivides the y-axis into horizontal rectangles.

28. **Give an example which shows that pointwise convergence does not preserve continuity in the limit.**
Answer: Let $f_n(x) = \{ \begin{matrix} 1 & x \geq \frac{1}{n} \\ nx & x \in (0, \frac{1}{n}) \\ 0 & x \leq 0 \end{matrix}$ so $f(x) = \{ \begin{matrix} 1 & x > 0 \\ 0 & x \leq 0 \end{matrix}$. Thus $\lim_{x \to 0}(\lim_{n \to \infty} f_n(x)) \neq f(0)$

29. **Give an example which shows that pointwise convergence does not preserve differentiability in the limit.**
Answer: Let $f_n : [-1, 1] \to \mathbb{R}$ for $f_n(x) = x^{\frac{2n}{2n-1}}$. Note that $f_n(x) = x \times x^{\frac{1}{2n-1}}$ and let $g_n(x) = x^{\frac{1}{2n-1}}$. Then $g_n \to g$ where on $[0, 1]$, $g(x) = \{ \begin{matrix} 0 & x = 0 \\ 1 & x \in (0, 1] \end{matrix}$ and on $[-1, 0]$, $g(x) = \{ \begin{matrix} 0 & x = 0 \\ -1 & x \in [-1, 0) \end{matrix}$.
Then $f_n \to f$ where $f(x) = \{ \begin{matrix} x & x \geq 0 \\ -x & x < 0 \end{matrix} = |x|$ which is not differentiable at $x = 0$.

30. **Give an example which shows that pointwise convergence does not preserve integrability in the limit.**
Answer: Let $f_n(x) = \{ \begin{matrix} 0 & x \in [q_1, ..., q_n] \\ 1 & else \end{matrix}$ where (q_n) is a sequence of rational numbers in $[0, 1]$. Then $f_n(x) \to f(x) = \{ \begin{matrix} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{matrix}$. Let P be a partition of $[0, 1]$ so that $m_i = 0$ and $M_i = 1$ which implies that $U(P, f) = 1$ and $L(P, f) = 0$ for any P. Thus f is not integrable because $1 = U(P, f) - L(P, f) < \epsilon$ does not hold for $\epsilon < 1$. But $f_1(x) = \{ \begin{matrix} 0 & x = q_1 \\ 1 & x \neq q_1 \end{matrix}$ is integrable with $\int_0^1 f_1(x)dx = 1$. f_2 is also integrable with $\int_0^1 f_2(x)dx = 1$. Thus $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 1$ but $\int_0^1 \lim_{n\to\infty} f_n(x)dx$ does not exist.

31. **If f_n is bounded for all n, is it true that $\lim_{n\to\infty} f_n(x)$ is bounded?**
Answer: No. Take $f_n(x) = \frac{n}{nx+1}$ which is bounded for all n. Then $\lim_{n\to\infty} f_n(x) = \frac{1}{x}$ which is unbounded at 0.

32. **If $f_n \to f$ converges uniformly and f_n' exists, does f' exists?**
Answer: No. Let $f_n(x) = \frac{sin(nx)}{\sqrt{n}}$ then $f_n(x) \to 0$ uniformly as $n \to \infty$. Then $f'(x) = 0$ and $f_n'(x) = \sqrt{n}cos(nx)$ and $f_n'(0) = \sqrt{n} \to \infty$ as $n \to \infty$. So uniform convergence does not preserve derivatives. i.e. $\lim_{n\to\infty}(\lim_{h\to 0} \frac{f_n(x+h)-f_n(x)}{h}) \neq \lim_{h\to 0}(\lim_{n\to\infty} \frac{f_n(x+h)-f_n(x)}{h})$.

33. **Explain the definition of the Riemann integral.**
Answer: The definition states "If $\inf U(P,f) = \sup L(P,f)$, then f is Riemann-integrable on $[a,b]$ where $U(P,f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$ and $L(P,f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$ where $M_i = \sup f(x)$ and $m_i = \inf f(x)$ for $x_i \leq x \leq x_i$." So m_i is the smallest value f takes on over the i^{th} interval while M_i is the largest value f takes on over the i^{th} interval and $(x_i - x_{i-1})$ is the length of the i^{th} interval. So $m_i(x_i - x_{i-1})$ and $M_i(x_i - x_{i-1})$ are area approximations with $m_i(x_i - x_{i-1}) \leq M_i(x_i - x_{i-1})$. By summing all of these area approximations, we get a upper and lower estimate for the area under the curve. Thus when the least upper bound of the lower estimate is equal to the greatest lower bound of the upper estimate, we say that f is Riemann integrable.

34. **Let f be continuous on $[a,b]$. Show that if $\int_a^c f(x)dx = 0$ for all $c \in [a,b]$, then $f = 0$ on $[a,b]$.**
Answer: Let $x_0 \in [a,b]$ and assume $f(x_0) \neq 0$. Let $\delta > 0$ then $f \neq 0$ in $B_\delta(x_0)$. Then $\int_a^{x_0+\delta} f(x)dx = 0$ and $\int_a^{x_0-\delta} f(x)dx = 0$ so
$\int_{x_0-\delta}^{x_0+\delta} f(x)dx = \int_a^{x_0+\delta} f(x)dx - \int_a^{x_0-\delta} f(x)dx = 0$.
But $\int_{x_0-\delta}^{x_0+\delta} f(x)dx > \frac{|f(x_0)|}{2} \times 2\delta > 0$ which is a contradiction so f is 0 everywhere.

35. **If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then is $f \in \mathscr{R}(\alpha)$?**
Answer: No, f must be continuous in some compact interval so that f is then uniformly continuous, then this statement would be true.

36. **We know that if $f \in \mathscr{R}(\alpha)$ and $g \in \mathscr{R}(\alpha)$ on $[a,b]$, then $fg \in \mathscr{R}(\alpha)$. Does $\int_a^b fg d\alpha = \int_a^b f d\alpha \int_a^b g d\alpha$?**
Answer: No. Let $f(x) = g(x) = x$, then $\int_a^b f(x)dx = \int_a^b g(x)dx = \frac{x^2}{2}|_a^b$. So $\int_a^b f(x)dx \int_a^b g(x)dx = b^2 - a^2$ but $\int_a^b f(x)g(x)dx = \frac{x^3}{3}|_b^a = \frac{b^3-a^3}{3}$.

37. **If $f_n \to f$ pointwise on $[a,b]$ and each f_n is integrable, then is f integrable such that $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$?**
Answer: No, for example let $f_n : [0,1] \to \mathbb{R}$ such that $f_n(x) = 2nx(1-x^2)^n$. For all $x \in [0,1]$, applying L'Hospital's Rule, we get $\lim_{n\to\infty} f_n(x) = 0$ which implies that $f(x) = 0$ so $\int_0^1 f = 0$. Then $\lim_{n\to\infty} \int_0^1 f_n = \lim_{n\to\infty} \int_0^1 2nx(1-x^2)^n dx$. Applying the Fundamental Theorem of Calculus and using u-substitution with $u = 1 - x^2$ and $du = -2x$, we get
$\lim_{n\to\infty} \int -nu^n du = \lim_{n\to\infty} \frac{-n}{n+1}(1-x^2)|_0^1 = \lim_{n\to\infty} \frac{n}{n+1} = 1$.

38. **Prove that** $\sup L(P, f) \leq \inf U(P, f)$**.**
Answer: Let $\inf U(P, f) = U(f)$ and $\sup L(P, f) = L(f)$. Suppose $U(f) < L(f)$. Then there exists a partition P_1 of $[a, b]$ such that $U(f) \leq U(P_1, f) \leq L(f)$ and there exists a partition P_2 of $[a, b]$ such that $U(f) \leq U(P_1, f) < L(f, P_2) \leq L(f)$ by definition of $U(f)$ and $L(f)$. But we know that $L(P_2, f) \leq U(P_1, f)$ which is a contradiction. Thus $\sup L(P, f) \leq \inf U(P, f)$.

39. **From Midterm Two: Let** $f : \mathbb{Q} \to \mathbb{R}$ **be a continuous map. Is it true that one can always find a continuous map** $g : \mathbb{R} \to \mathbb{R}$ **extending f, namely,** $g(x) = f(x)$ **for any** $x \in \mathbb{Q}$**?**
Answer: No, this is is not true. Let $f : \mathbb{Q} \to \mathbb{R}$ with $f(x) = \{ \begin{array}{ll} -1 & x < \sqrt{2}, x \in \mathbb{Q} \\ 1 & x > \sqrt{2}, x \in \mathbb{Q} \end{array}$.
By definition, $\mathbb{Q}$ and $\emptyset$ are open in $\mathbb{Q}$ and $(-\infty, \sqrt{2}) \cap \mathbb{Q}$ and $(\sqrt{2}, \infty) \cap \mathbb{Q}$ are also open in $\mathbb{Q}$ and these are the only possible preimages $f^{-1}(E)$ for $E \subset \mathbb{R}$. Suppose there exists a continuous extension g to $\mathbb{R}$, then the left and right limit at $\sqrt{2}$ are $g(\sqrt{2}+) = f(\sqrt{2}+) = 1$ and $g(\sqrt{2}-) = f(\sqrt{2}-) = -1$ which contradicts that $g(\sqrt{2}-) = g(\sqrt{2}+)$ so there is no continuous extension.

40. **What is the difference between a topological space and a metric space?**
Answer: A metric space is a specific type of topological space where the notion of distance is defined. A topological space has a topology which "is a family of subsets that is closed under arbitrary unions and finite intersections" (https://link.springer.com/content/pdf/bbm%3A978-1-4614-1891-7%2F1.pdf).

41. **Does compactness imply completeness of a metric space?**
Answer: Let (X, d) be a metric space and let (x_n) be a Cauchy sequence in (X, d). Let x_{n_k} be a subsequence that converges to $x \in (X, d)$. Since $x_{n_k} \to x$ then there exists N_1 such that $n_k \geq N_1$ implies $|x_{n_k} - x| < \frac{\epsilon}{2}$. Let N_2 be such that $n, m \geq N_2$ implies $|x_n - x_m| < \frac{\epsilon}{2}$. Let $n > N$ with $N = max\{N_1, N_2\}$. Then $|x_n - x| = |x_n - x_N + x_N - x| \geq |x_n - x_N| + |x_N - x| < \epsilon$. So (X, d) is complete.

42. **What does it mean for a set to be dense in a metric space?**
Answer: A set $S \subset X$ is dense in X if, for any $\epsilon > 0$ and $x \in X$, there is some $|x - s| < \epsilon$. (https://brilliant.org/wiki/dense-set/)

43. **(From UC Berkeley Past Exam Archive from Math Department) Let** $g : [a, b] \to \mathbb{R}$ **be one-to-one and differentiable on the set** (a, b)**. Show** $\int_a^b g(x)dx + \int_{g(a)}^{g(b)} g^{-1}(u)du = bg(b) - ag(a)$**.**
Answer: So $g([a, b])$ is an interval and g^{-1} is differentiable. $\int_a^b (g^{-1} \circ g(x))g'(x)dx = \int_{g(a)}^{g(b)} g^{-1}(u)du$ with $f = g^{-1}$ and $u = g(x)$. But $(g^{-1} \circ g)(x) = x$ for all $x \in [a, b]$. So $\int_{g(a)}^{g(b)} g^{-1}(u)du = \int_a^b xg'(x)dx$. Then

by integration by parts, $\int_{g(a)}^{g(b)} g^{-1}(u)du = xg(x)|_{x=a}^{b} - \int_a^b g(x)dx$. Thus $\int_a^b g(x)dx + \int_{g(a)}^{g(b)} g^{-1}(u)du = bg(b) - ag(a)$.

44. **(From Professor Hass's Practice Exam) Let f be integrable on $[0,1]$ and (x_n) be the sequence $x_n = 1 - \frac{1}{n^2}$ for all $n \in \mathbb{N}$. Show $\int_0^1 f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} \int_{x_k}^{x_k+1} f(t)dt$.**
Answer: $\sum_{k=1}^{n} \int_{x_k-1}^{x_k} f(x)dx = \int_{x_0}^{x_n} f(x)dx = F(x_n)$ which is a constant (because x_n is increasing to 1).
Thus $\lim_{x_n\to 1} \int_{x_0}^{x_n} f(x)dx = F(1) = \int_{x_0}^{1} f(x)dx$

45. **Prove that if (a_n) is a sequence in $\mathbb{R}$, then $\limsup_{n\to\infty}(n|a_n|)^{\frac{1}{n}} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$.**
Answer: Let $\epsilon > 0$. We know that $n^{\frac{1}{n}} \to 1$ as $n \to \infty$. Then there exists $N \in \mathbb{N}$ such that $n > N$ implies $1 - \epsilon < n^{\frac{1}{n}} < 1 + \epsilon$, then
$(1-\epsilon)|a_n|^{\frac{1}{n}} < n|a_n|^{\frac{1}{n}} < (1+\epsilon)|a_n|^{\frac{1}{n}}$ for n-large. Then
$\limsup(1-\epsilon)|a_n|^{\frac{1}{n}} \le \limsup n^{\frac{1}{n}}|a_n|^{\frac{1}{n}} \le \limsup(1+\epsilon)|a_n|^{\frac{1}{n}}$. So
$(1-\epsilon)\limsup|a_n|^{\frac{1}{n}} \le \limsup n^{\frac{1}{n}}|a_n|^{\frac{1}{n}} \le (1+\epsilon)\limsup|a_n|^{\frac{1}{n}}$. Since $\epsilon > 0$,
$\limsup|a_n|^{\frac{1}{n}} = \limsup(n|a_n|)^{\frac{1}{n}}$.

46. **What is radius of convergence?**
Answer: The radius of convergence is the radius of the largest disk centered at some point such that the series converges for all values inside the disk.

47. **(From UC Berkeley Past Exam Archive from Math Department) Find the radius of convergence of $S(x) = \sum_{n=1}^{\infty} \frac{(x+4)^n}{n}$ for $x_0 = -2$.**
Answer: $\lim_{n\to\infty} \frac{|x+4|}{n^{\frac{1}{n}}} = |x+4| \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}}}$. If $|x+4| < 1$, S exists, if $|x+4| > 1$, S does not exist. $|x+4| < 1$ holds if and only if $x \in B_1(-4)$ so R=1.

48. **Find the radius of convergence of $S(x) = \sum_{n=0}^{\infty} n!x^n$.**
Answer: $\lim_{n\to\infty} \frac{|a_{n+1}x^{n+1}|}{|a_n x^x|} = |x| \lim_{n\to\infty} \frac{(n+1)!}{n!} = |x| \lim_{n\to\infty}(n+1) = \{ \begin{matrix} \infty & x \neq 0 \\ 0 & x = 0 \end{matrix}$. Thus $R = 0$.

49. **When is Heine-Borel Theorem true?**
Answer: Heine-Borel Theorem only holds in $\mathbb{R}^n$. (e.g. As we saw previously, this is not necessarily true in $\mathbb{Q}$).

50. **Prove that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.**
Answer: Because $\ln x$ is continuous, $\ln(\lim_{n\to\infty} n^{\frac{1}{n}}) = \lim_{n\to\infty} \frac{\ln n}{n} = 0$. Then $\lim_{n\to\infty} n^{\frac{1}{n}} = e^0 = 1$.