Math 104 Final Review:

Midterm 1

1. Completeness axiom of real numbers and limit of sequence of real numbers: (Ross 1.1, 1.2, 1.3, 1.4)

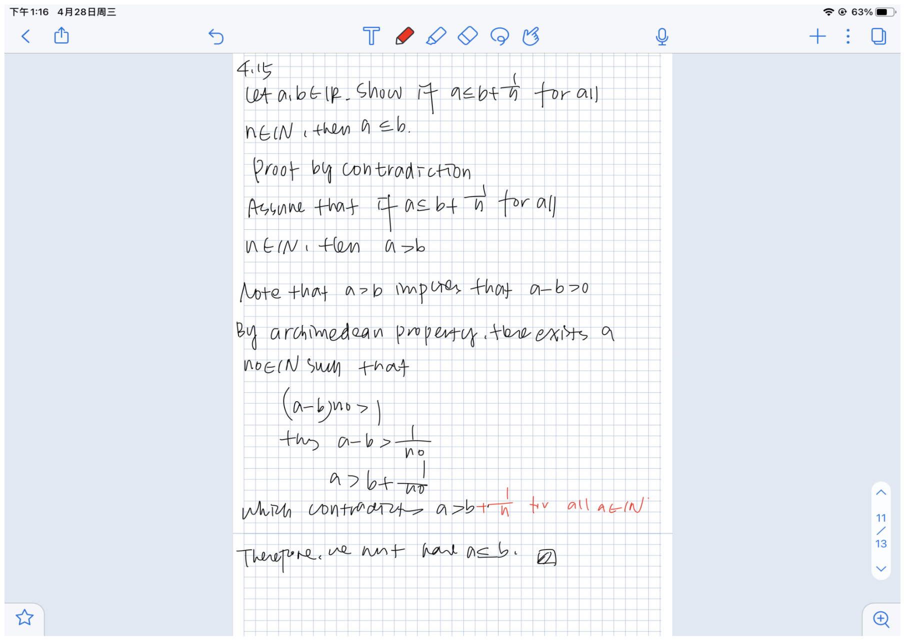
Lecture 1:

- **Natural Numbers** N = {1, 2, 3, . . . }
- **Integers** Z = {. . . , −3, −2, −1, 0, 1, 2, 3, . . . }
- **Rational Numbers (ratios of integers)** Q = {p/q : p, q ∈ Z and q≠0}
- **Complex Numbers** C = {x + iy : x, y ∈ R}
- **Propositions**: if r is a rational number, (r=c/d, gcd(c,d)=1) and it's a root of the following integer coefficients polynomial: Cnx^n+Cn−1x^n−1+…+C0=0, then d divides *Cn, c divides C*0
- **Corollary**: if r =c/d ≠0 is a root of a "monic polynomial", its lead term has coefficient is 1: x^n+Cn−1x^n−1+…+C0=0 (we can apply this to prove that the roots of x^2-2=0 are not rational numbers)

Lecture 2:

- **Supremum and Infimum**: Let S be a non-empty subset of R. If S is bounded above (i.e exists an upper bound for S), and S has a least upper bound, then we call it the supremum of S denoted as sup(S). If S is bounded below and S has a greatest lower bound, then we call it the infimum of S, denoted as inf (S).
  NOTE: the difference between sup, inf and max, min, for example, S=(0,1), max(S) doesn't exist, least upper bound(S)=1, greatest lower bound(S)=0
- **Completeness Axiom**: Every non-empty subset S ⊂ R that is bounded above has a least upper bound, and if S is bounded from below, then inf S exists.
- **Archimedean Property**: If a>0,b>0 are real numbers, then for n∈ N, we have an>b
- **Sequence**: a1,a2,a3,…,an,∈R, could be denoted as (an)n∈N
  NOTE: sequence is not a set. Sequence has its element come in order (first, second…). Set is just information about "who is in the set".
- **Some important propositions about inf and sup**:
  1. Suppose that A, B are nonempty sets of real numbers such that x ≤ y for all x ∈ A and y ∈ B. Then sup A ≤ inf B
  2. Suppose that A, B are subsets of R such that A ⊂ B. Then sup A ≤ sup B, and inf A ≥ inf B.
  3. If A ⊂ R, then M = sup A if and only if: (a) M is an upper bound of A; (b) for every M0 < M there exists x ∈ A such that x > M0 . Similarly, m = inf A if and only if: (a) m is a lower bound of A; (b) for every m0 > m there exists x ∈ A such that x < m0 .

Homework 1:

- 

4.15

Let $a, b \in \mathbb{R}$. Show if $a \le b + \frac{1}{n}$ for all $n \in \mathbb{N}$, then $a \le b$.

Proof by contradiction

Assume that if $a \le b + \frac{1}{n}$ for all $n \in \mathbb{N}$, then $a > b$

Note that $a > b$ implies that $a - b > 0$

By archimedean property, there exists a $n_0 \in \mathbb{N}$ such that

$(a-b)n_0 > 1$

thus $a - b > \frac{1}{n_0}$

$a > b + \frac{1}{n_0}$

which contradicts $a \le b + \frac{1}{n}$ for all $n \in \mathbb{N}$

Therefore, we must have $a \le b$. ☐

2. Limit of bounded monotonic sequence, limit theorem, liminf, limsup (Ross 2.7,2.9,2.10)

Lecture 3:

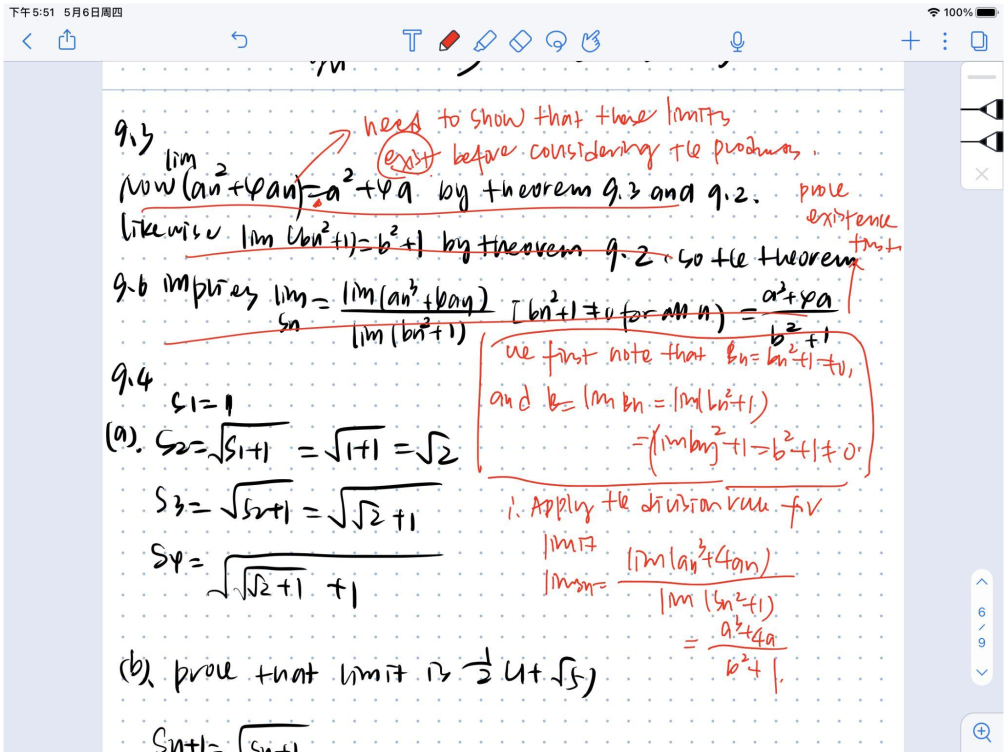
- **Limit of Sequence**: We say a sequence (an)n has limit $\alpha \in$ R, if $\forall \epsilon > 0$, there exists a real number N>0, such that n>N, we have $|a_n - \alpha| < \epsilon$, denoted by $\lim_{n\to\infty} an = \alpha$
- **Property and tools to find limit**:
  1. All convergent sequences are bounded
  2. If liman=α, and if k$\in$R, then lim(k.an)=k.α
  3. Let an, bn be convergent sequences, liman=α,limbn=β, then
     a. lim(an+bn)=lim(an)+lim(bn)=α+β
     b. lim(an.bn)=(liman).(limbn)=αβ
     c. If an≠0, $\forall$ n and if $\alpha \neq 0$, then, lim(1/an)=1/α

Lecture 4:

- **Continued important theorems:**
  1. $\lim_{n\to\infty} 1/n$^p=0 $\forall$ p>0
  2. lima^n=0, if $|a|<1$
  3. $\lim_{n\to\infty}$n^(1/n)=1
  4. $\lim_{n\to\infty}$a^1/n=1 for a>0
- **Cauchy Sequence:** if $\forall \epsilon > 0$, there exists N>0, s.t $\forall$ n1,n2>N, we have $|a_{n1} - an2| < \epsilon$, (using graph--oscillation amplitude gets smaller and smaller)
- **Monotone Sequence**: an increasing sequence is such that an+1≥an, a decreasing sequence is such that an+1≤an, they are both called monotone sequences.

- **Theorems**:
    1. All bounded sequences are convergent
    2. Let (an) be a sequence, (an) is cauchy $\Leftrightarrow a_n$ converges

Homework 2



3. Subsequences and Cauchy Sequences: (Ross 2.10, 2.11)

Lecture 5

- **Monotone Sequence and Cauchy**:
    1. limsup: Let (a_n) be a sequence in R, $\limsup a_n=\lim_{N\to\infty}(\sup_{n>N} a_n)$
    2. liminf :Let (a_n) be a sequence in R, $\liminf a_n=\lim_{N\to\infty}(\inf_{n>N} a_n)$

Remark: if we allow the notion of lim=+∞ or lim=-∞, then limsup exists in R or =-∞, liminf exists in R or =+∞

3. A monotone increasing sequence has several properties;
    a. If it's bounded, then its limit exists
    b. If it's unbounded (without upper/lower bounds), then $\lim a_n=+\infty$
    c. Strategy: if we want to prove a=b, one way to prove it is $|b-a|<\epsilon$, for any $\epsilon>0$

- **Lemma:**

a. if (an) is bounded sequence, then its limsup, liminf exists

b. If (an) is bounded sequence, and $\alpha_+=\limsup a_n$, then for any $\epsilon>0$, $\exists N$ such that $\forall n>N$, we have $a_n<\alpha_++\epsilon$

- **Theorem:** let (an) be a bounded sequence. Then lim an exists⟺limsup(an)=liminf(an)

Lecture 6

- **Subsequence:** let (Sn) be a sequence of real numbers. Given a strictly increasing sequence of indices $n1<n2<n3<...nm<$....we define the corresponding subsequence as t_m:=Snm, (t_m) is called a subsequence of (sn).Sometimes, we write (Snk)k for the subsequence.
- **Lemma:**
- a. if (Sn) is convergent, then any subsequence converges to the same point
- b. If α=limSn exists in R, then there exists a subsequence that is also monotone.
- c. let (Sn) be any sequence. Then for any t∈ R, (Sn) has a subsequence converges to t ⟺ $\forall\, \epsilon$>0, the set $\{n \in$ N : $|Sn$-t$|<\epsilon\}$ is infinite.

Lecture 7

- **Theorem:** (Bolzano-Weierstrass Theorem)
  a. Every bounded sequence has a convergence subsequence.
  b. Let (Sn) be a sequence, S be the set of subsequential limits of (Sn), then
     1. S in nonempty
     2. supS=limsup(Sn), infS=liminf(Sn)
     3. S=$\{\alpha\}$⟺ limSn exists and equals to α

  c. Let S be the set of subsequential limits of (sn). Suppose (tn) is a sequence in S∩R and t=lim(tn). Then t∈S

- **Subsequential limit:** let (Sn) be a sequence in R. A subsequential limit is any real number or the +∞,-∞, that is the limit of a subsequence
- **Lemma:** let (Sn) be any sequence. Then there exists a monotone sequence whose limit is limsup(Sn). Similarly, there exists a monotone sequence whose limit is liminf(Sn)

Lecture 8

- **Themorem:**
  a. let sn be a sequence with limit S>0, let tn be any sequence. Then limsup (sn.tn)=S. limsup(tn)
  b. Let (Sn) be a sequence of positive numbers, then we have liminf (Sn+1/Sn)≤liminf(Sn)^(1/n)≤limsup(Sn)^(1/n))≤limsup(Sn+1/Sn)

Homework 3

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1.

(a) We define $\limsup\{s_n, s_{n+1}, \dots\} = \limsup s_n$

$\liminf\{s_n, s_{n+1} \dots\} = \liminf s_n$

we know that $\inf\{s_n, s_{n+1} \dots\} \le \sup\{s_n, s_{n+1} \dots\}$ (a)

for each $n \in \mathbb{N}$.

Monotone convergence says that both $\inf\{s_n, s_{n+1} \dots\}$ and $\sup\{s_n, s_{n+1} \dots\}$ converge. since they are bounded and are monotone.

$\sup\{s_n, s_{n+1} \dots\} - \inf\{s_n, s_{n+1} \dots\} \ge 0$ for each $n \in \mathbb{N}$.

so $\liminf s_n \le \limsup s_n$

Let $A_N = \sup\{s_n \mid n \ge N\}$

$B_N = \inf\{s_n \mid n \ge N\}$

Then $A_N \ge B_N$.

Since (s_n) is bounded.

hence $\lim A_N$ and $\lim B_N$ exists. Hence

$\lim A_N \ge \lim B_N$

thus $\limsup s_n \ge \liminf s_n$

(b)

Since $A_N \ge A_{N+1}$ and A_N is a bounded sequence,

hence $\lim A_N = \inf\{A_N \mid N \in \mathbb{N}\}$.

we can also prove it directly. let $u = \inf\{N \mid N \in \mathbb{N}\}$.

Then $A_N \ge u$, $\forall N$. And for any $\varepsilon > 0$. $\exists N$, s.t.

$u + \varepsilon > A_N$, otherwise $u + \varepsilon$ would be a bigger lower bound of $\{A_N\}$. contradict the def of inf.

$\Rightarrow$ $\forall n > N$, we have

$u + \varepsilon > A_N \ge A_n \ge u \Rightarrow |A_n - u| < \varepsilon$ ✓

Hence, A_n converges to u

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2.

Define $A_k = \sup\{a_n : n \ge k\}$, $B_k = \sup\{b_n : n \ge k\}$, so

by definition $\limsup\limits_{n\to\infty} a_n = \lim_{k\to\infty} A_k$ and similarly

for B_k: $\limsup_{n\to\infty} b_n$

we consider $\lim\limits_{k\to\infty} C_k = \limsup_{n\to\infty}(a_n + b_n)$

then for all $n \ge k$, we have $a_n + b_n \le A_k + B_k$, because we estimate a_n by the supremum of all terms of (a_n) with $n \ge k$ and likewise for the b_n.

$C_k = \sup\{(a_n + b_n : n \ge k\} \le A_k + B_k$

This holds for all k, so we take lim on both sides as $k \to \infty$.

$\lim\limits_{k\to\infty} C_k = \lim\limits_{k\to\infty}(A_k) + \lim\limits_{k\to\infty}(B_k)$

Thus $\limsup\limits_{n\to\infty}(a_n + b_n) = \limsup\limits_{n\to\infty} a_n + \limsup\limits_{n\to\infty} b_n$ ◻

$a_n = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$ $b_n = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd.} \end{cases}$

$a_n + b_n = 1$

$\therefore \limsup(a_n + b_n) = 1$

$\limsup(a_n) = \limsup(b_n) = 1$

e.g. a_n and b_n be sequences repeat in cycle of 4

$a_n = (0, 1, 2, 1, 0, 1, 2, 1 \dots)$ $b_n = (2, 1, 2, 0, 2, 1, 2, 0 \dots)$

$a_n + b_n = (2, 3, 4, 1, 2, 3, 4, 1 \dots)$ ↳ not a strict example.

$\limsup(a_n + b_n) = 4$

$\limsup a_n = 2$ $\limsup b_n = 2$ $\therefore \limsup a_n + \limsup b_n = 2 + 2 = 4$

$\therefore \limsup(a_n + b_n) = \limsup a_n + \limsup b_n$.

an example that $\limsup(a_n + b_n) < \limsup a_n + \limsup b_n$.

$|s_{n+1}-s_n| = \frac{1}{n+1} < \frac{1}{n}$ for each n, but (s_n) is not bounded.

so (s_n) is not a Cauchy sequence and thus not convergent

10.7 ✗

Let $u = \sup S$, for each $\varepsilon > 0$, $\exists s \in S$, s.t. $u - \varepsilon < s \le u$

Hence, $\forall n \in \mathbb{N}$, let $\varepsilon = \frac{1}{n}$, we can choose $s_n \in S$

s.t.

$u - \frac{1}{n} < s_n \le u$. Thus (s_n) forms a sequence in S convergent to u

10.8 $\sigma_n = \frac{1}{n}(s_1+s_2+\cdots+s_n)$. prove (σ_n) is an increasing sequence.

(s_n) be sequence of positive numbers

7\. let (s_{n_k}) converges to t. Then $\forall \varepsilon > 0$, $\exists K > 0$

s.t. $|s_{n_k} - t| < \varepsilon$, $\forall k > K$

then $\{n_k \mid k > K\}$ is an infinite set

contained in $\{n \mid |s_n - t| < \varepsilon\}$

$\Leftarrow$ We are going to build n_k iteratively

let $\varepsilon_k = \frac{1}{k}$ for $k = 1, 2, 3, \ldots$

Then let $n_1 \in \{n \mid |s_n - t| < \varepsilon_1\}$

Assume $n_1 \ldots n_{k-1}$ are constructed, s.t.

$n_1 < n_2 < \cdots n_{k-1}$ $\quad |s_{n_i} - t| < \varepsilon_i$

We can construct n_k as follo

pick any $n_k \in \{n \mid n > n_{k-1}, |s_n - t| < \varepsilon_k\}$

By induction, we get a seq $n_1 < n_2 < n_3 < \cdots$

Since $|s_{n_k} - t| < \varepsilon_k = \frac{1}{k}$. We have $s_{n_k} \to t$ as $k \to \infty$ ☐

If $t \in S$, then the constant sequence $s_n = t$ would work. If $t \notin S$, then let $t = 0.b_1 b_2 \ldots$ be a decimal expansion of t. (if the expansion is finite we add trailing zeros). Then define $s_n = 0.b_1 \ldots b_n 3$.

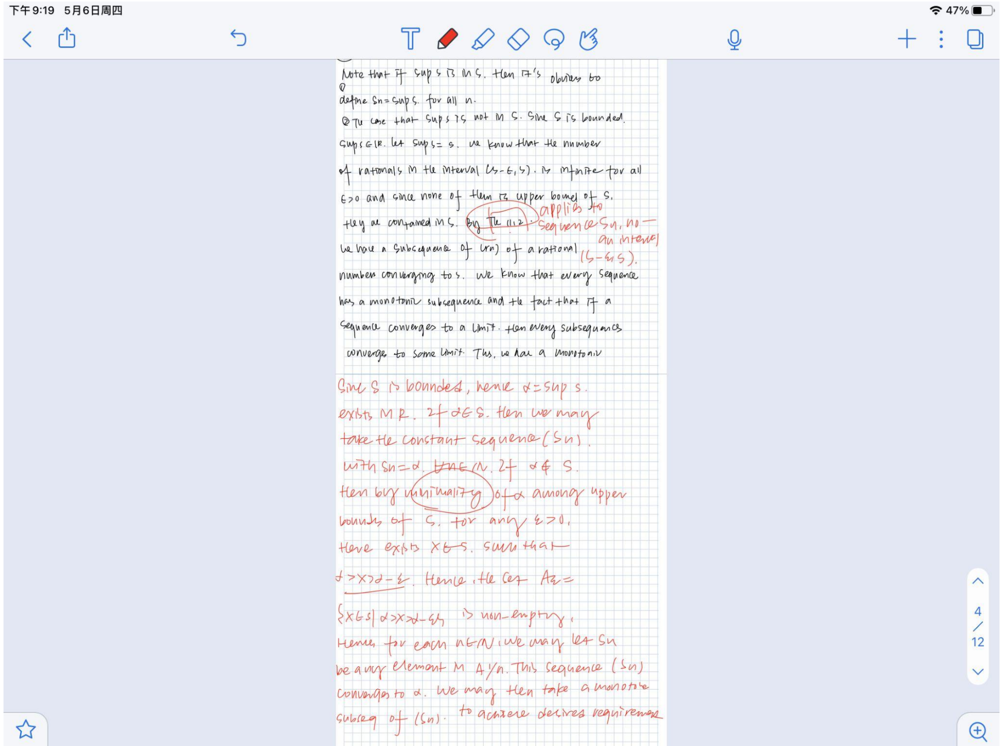
$\therefore |s_n - t| = |0.\underbrace{0\ldots0}_{n \text{ many}}3 - 0.0\ldots0 b_{n+1} \ldots|$

$\le |0.0\ldots03| + |0.0\ldots0 b_{n+1} b_{n+2} \ldots|$

$\le 10^{-n} + 10^{-n} = 2 \cdot 10^{-n}$

Since $2 \cdot 10^{-n} \to 0$ as $n \to \infty$, we have $s_n \to t$.

Homework 4



Note that if sup S is in S, then it's obvious to
① define Sn = sup S for all n.
② The case that sup S is not in S. Since S is bounded,
sup S ∈ ℝ. let sup S = s. we know that the number
of rationals in the interval (s−ε, s) is infinite for all
ε>0 and since none of them is upper bound of S,
they are contained in S. By Thm 11.2 applies to sequence Sn, no— an interval (s−ε, s).
we have a subsequence of (rn) of a rational
number converging to s. we know that every sequence
has a monotonic subsequence and the fact that if a
sequence converges to a limit, then every subsequences
converge to some limit. Thus, we have a monotonic

Since S is bounded, hence α = sup S
exists in ℝ. If α ∈ S, then we may
take the constant sequence (Sn),
with Sn = α ~~∀n∈ℕ~~. If α ∉ S,
then by minimality of α among upper
bounds of S, for any ε>0,
there exists x∈S, such that
α>x>α−ε. Hence, the set Aε =
{x∈S | α>x>α−ε} is non-empty.
Hence, for each n∈ℕ, we may let Sn
be any element in A1/n. This sequence (Sn)
converges to α. We may then take a monotone
subseq of (Sn) to achieve desired requirement

4. Metric Space and Topology (Ross 2.13)

Lecture 9

- **<u>Metric Space</u>**: A metric space is a set S, together with a distance function d:S×S→R, such that 1. d(x,y)≥0, and d(x,y)=0⇔x=y
  2. d(x,y)=d(y,x)
  3. d(x,y)+d(y,z))≥d(x,z)
- **<u>Cauchy Sequence in a metric space (S,d):</u>** A sequence (Sn) in S is cauchy if ∀ϵ>0, there exists a N>0, such that ∀n,m>N, d(Sn,Sm)<ϵ
- **<u>Convergence in a metric space (S,d):</u>** A metric space (S,d) is complete, if every Cauchy sequence is convergent
- **<u>Induced distance function:</u>** if (S,d) is a metric space, and A⊂S is any subset, then (A,d|A×A)is a metric space
- **<u>Completeness:</u>** A metric space (S,d) is complete if every Cauchy sequence is convergent.
- **<u>Bolzano-Weierstrass Theorem for R^n:</u>** Every bounded sequence (sm)m∈R^n has a convergent subsequence.
- **<u>Topology:</u>** let S be a Set. A topological structure on S is the data of a collection i of subset S, if U⊂S, and U∈I

Homework 5

- Prove well-defined first

Math 104 HW 5
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Ex 13.3 13.4 13.6 13.7

Since $\{x_n\}$ and $\{y_n\}$ bounded thus $(x_n - y_n)$ is a bounded seq. hence $\sup\{|x_n - y_n|\}$ exists in $\mathbb{R}$. Hence $d(x,y) \in \mathbb{R}$. $\forall (x_n)(y_n)$ in B

13.3 Let B be the set of all bounded sequences $x = (x_1, x_2, \ldots)$ and define $d(x,y) = \sup\{|x_j - y_j| : j = 1, 2 \cdots\}$

(a). Show d is a metric for B

* need to show that d is well-defined

Check

(i) $d(x,x) = \sup\{|x_j - x_j| : j = 1, 2 \cdots\} = 0$. for all $x \in S$

$d(x,y) = \sup\{|x_j - y_j| : j = 1, 2, \cdots\} > 0$

(ii) $d(x,y) = \sup\{|x_j - y_j| : j = 1, 2 \cdots\} = \sup\{|y_j - x_j| : j = 1, 2, \cdots\} = d(y,x)$ for all $x, y \in S$

(iii) $|x_j - z_j| \le |x_j - y_j| + |y_j - z_j|$.

$\therefore \sup\{|x_j - z_j| : j = 1, 2 \cdots\} \le \sup\{|x_j - y_j| : j = 1, 2 \cdots\} + \sup\{|y_j - z_j| : j = 1, 2 \cdots\}$

$\therefore d(x,z) \le d(x,y) + d(y,z)$ for all $x, y, z \in S$

(b). (i) $d^*(x,x) = \sum_{j=1}^{\infty} |x_j - x_j| = 0$ for all $x \in S$.

$d^*(x,y) = \sum_{j=1}^{\infty} |x_j - y_j| = |x_1 - y_1| + |x_2 - y_2| + \cdots |x_j - y_j| > 0$

13.4 Prove (iii) and (iv) in Discussion 13.7

assume include 3 open sets

(iii). Union of a collection of open set is open

U be a collection of open sets ~~(U_1, U_2, U_3)~~.

$X = \bigcup_{i \in \mathbb{N}} (U_i)$.

① Let $\{U_\alpha\}_{\alpha \in A}$ be a collection of open sets. If $p \in \bigcup_\alpha U_\alpha$, then $\exists\, \alpha_0 \in A$ s.t. $p \in U_{\alpha_0}$. Hence, $\exists\, \delta > 0$. $B_\delta(p) \subset U_{\alpha_0}$, since U_{α_0} is open. $\therefore B_\delta(p) \subset U_{\alpha_0} \subset \bigcup_\alpha U_{\alpha_0}$.

Let $x \in \bigcup_{i \in \mathbb{N}} (U_i)$. Then $x \in U_i$ for some $i \in \mathbb{N}$.

$\because U_i$ is open. $r > 0$

$\therefore \{s \in U_i : d(x,s) < r\} \subseteq U_i \subseteq \bigcup_{i \in \mathbb{N}} (U_i)$

Therefore $\bigcup_{i \in \mathbb{N}} (U_i)$ is open

(iv). The intersection of finitely many open set is again, an open set.

$A = \bigcap_{i=1}^{k} U_i$, $k \in \mathbb{N}$. U_i are open sets

Let $x \in \bigcap_{i=1}^{k} U_i$. Then $x \in U_i$ for $i \in \{1, 2 \cdots k\}$.

For each U_i. there exist r_i such that.

$\{s \in U_i . d(s,x) < r_i\} \subseteq U_i$

Let $r = \min\{r_1 \cdots r_k\}$.

$\{s : d(x,s) < r\} \subseteq \{s : d(x,s) < r_i\}$ for $i \in \{1, 2, \cdots k\}$

② If $p \in \bigcap_{i=1}^{N} U_i$. U_i open, then $\exists\, \delta_i > 0$. $\forall i \in \{1 \cdots N\}$ s.t. $B_{\delta_i}(p) \subset U_i$. Let $\delta = \min\{\delta_i\}$. Then $B_\delta(p) \subset U_i$ $\forall i \Rightarrow B_\delta(p) \subset \bigcap_{i=1}^{N} U_i$

$\therefore \{s : d(x,s) < r\} \subseteq U_i$ for all $i \in \{1, 2, \cdots k\}$

implies $\{s : d(x,s) < r\} \subseteq \bigcap_{i=1}^{k} U_i$.

Therefore $\bigcap_{i=1}^{k} U_i$ is an open set.

13.6 Prove proposition 13.9

⟹ $E^- \supset E$ by definition, and $E^- = \cap F$ for all closed $F \supset E$. and we can take one of the $F = E$. hence $E^- \subset E$.

(a). The set E is closed if and only if $E = E^-$.

E^- is the closure of E

∴ it's trivial to prove that $E \subseteq E^-$, since E^- is defined the intersection of all of the closed set containing E.

∵ E is a closed set, it contains E and one the sets in the intersection

∴ $E^- \subseteq E$ ⟹ need to prove

Therefore $E^- = E$

If $E = E^-$, then E is closed. This shows $E^- = E$

(b). The set is closed if and only if it contains the limit of every convergent sequence of points in E.

⟸ E^- is an intersection of closed sets, hence closed

⟹ E is closed. Proof by contradiction.

Lecture 10

Midterm 1