Math 104 Review Notes

Ross

Chp 1

The natural numbers or $\mathbb{N}$, is all positive integers $\{1, 2, 3, \ldots\}$

Each n in $\mathbb{N}$ has a sucessor $n+1$

N1) 1 belongs to $\mathbb{N}$

N2) if $n \in \mathbb{N}$, then $n+1 \in \mathbb{N}$

N3) 1 is not the sucessor to any element in $\mathbb{N}$

N4) if $n+1 = m+1$ then $n = m$

$\mathbb{N}$ is the basis of mathematical induction

Q's:

Why aren't other sets like $\mathbb{Q}$ or $\mathbb{R}$ used for mathematical induction?

§2

$\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$

introducing division into integers calls for a new set of numbers namely an integer over an integer - $\frac{n}{m}$

$\sqrt{2}$ is not a rational, but can be approximated by rational numbers

- π and e are not rational

2.1 Def: A number is called an algebraic number if it satisfies a polynomial equation

$$c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0 = 0 \quad , \quad c_1, c_2, \ldots c_n \in \mathbb{Z}$$

- rational numbers are always algebraic numbers

if $r = \frac{m}{n}$ is a rational number then it satisfies the equation $nx - m = 0$

Q's: Can a rational number not be an algebraic number

Why aren't $\sqrt{2}, \pi, e$ not rational?

is there a set of numbers $\frac{n}{m}$ where $n, m \in \mathbb{N}$?

Rational Zeros theorem:

Suppose $c_0, c_1, \ldots, c_n$ are integers and r is a rational number satisfying the polynomial equation

$$c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0 = 0$$

where $n \geq 1$, $c_n \neq 0$ and $c_0 \neq 0$. Let $r = \frac{c}{d}$ where $c, d \in \mathbb{Z}$ having no common factors and $n \neq 0$. Then c divides c_0 and d divides c_n.

Q's:

How would r satisfy that equation?

§3

The set $\mathbb{R}$, the real numbers, includes all rational numbers, algebraic numbers, π, e, and $\sqrt{2}$.

if $a, b \in \mathbb{R}$, then

$a + b \in \mathbb{R}$

$a \times b \in \mathbb{R}$

Thm 3.2

(i) if $a \leq b$ then $-b \leq -a$

(ii) if $a \leq b$ then $c \leq 0$ then $bc \leq ac$

(iii) if $a \leq 0$ an $0 \leq b$ then $0 \leq ab$

(iv) $0 \leq a^2$ for all a

(v) $0 \leq 1$

(vi) if $0 < a$, then $0 < a^{-1}$

(vii) if $0 < a < b$ then $0 < b^{-1} < a^{-1}$, for $a, b, c \in \mathbb{R}$

3.4 Def: For number a and b we define $\text{dist}(a,b) = |a-b|$

$\text{dist}(a,b)$ represents the distance between a and b.

3.6 Corollary

$dist(a,c) \le dist(a,b) + dist(b,c)$ for all $a,b,c \in \mathbb{R}$

3.7 Tringle Inequality

$|a+b| \le |a| + |b|$ for all a,b.

Q's:

- How do we know they aren't any gaps in $\mathbb{R}$?
- Does the tringle inequality hold in abstract planes?
- How are vectors defined in $\mathbb{R}$?

§4 The Completeness Axiom

This axiom assures there are not gaps in $\mathbb{R}$.

4.1 Def:

Let S be a nonempty subset of $\mathbb{R}$

(a) if S contains a largest element s_0 (that is s_0 belongs to S and $s \le s_0$ for all $s \in S$), then we call s_0 the maximum of S: $s_0 = \max S$

(b) if S contains a smallest element, then we call the smallest element the minimum of S: $s_1 = \min S$

4.2 Def:

Let S be a nonempty set of $\mathbb{R}$

(a) if a real number M satisfies $s \le M$ for all $s \in S$, then M is called an upper bound of S and the set is said to be bounded above

(2) if a real number m satisfies $s \ge m$ for all $s \in S$ then m is a lower bound of S, and S is bounded below.

(c) the set S is bounded if it is bounded above and below. So S is bounded and there exists real numbers m and M s.t. $S \in [m, M]$

4.3 Def:

Let S be a nonempty subset of $\mathbb{R}$

(a) if S is bounded above and S has a least upper bound, then the least upper bound is the supremum of S, denoted by $\sup S$

(b) if S is bounded below and S has a greatest lower bound, then the greatest lower bound is the infimum of S, denoted by $\inf S$

Unlike max and min, sup and inf don't need to belong to S, i.e. $\sup, \inf \notin S$

. if S is bounded above, then $M = \sup S$, iff

(i) $s \leq M$ for all $s \in S$

(ii) whenever $M_1 < M$, there exists $s_1 \in S$ s.t. $s_1 > M_1$

4.4 Completeness Axiom

Every non-empty subset S of $\mathbb{R}$ that is bounded above has a least upper bound, i.e. $\sup S$ exists and is a real number

- Completeness Axiom doesn't hold for $\mathbb{Q}$

4.5 Corollary:

Every non-empty subset S of $\mathbb{R}$ that is bounded below has a greatest lower bound, $\inf S$

Can we write $\mathbb{R} \cup \{-\infty, +\infty\} = \mathbb{R}$?
What properties of $\mathbb{R}$ hold for $-\infty, \infty$?
Why is $[a, \infty)$ closed?

4.6 Archimedean Property
if $a>0$ and $b>0$, then for some positive inter n, we have $na>b$

- tells that if a is really small and b is really big, you can multiply a by some integer to get $a>b$.

4.7 Denseness of $\mathbb{Q}$
if $a, b \in \mathbb{R}$ and $a<b$, then there is a $r \in \mathbb{Q}$ s.t. $a<r<b$

Q's:
- Are ∞ and $-\infty$ bounds for infinite sets?
- Does the compleness Axiom hold for $\mathbb{C}$? As $\mathbb{R} \subset \mathbb{Q}$
- Can the sup and inf be elemnts of S? ie. sup, inf $\in S$
- Do sup and inf work on any plane?

§5 The symbols $+\infty$ and $-\infty$
$+\infty$, $-\infty$ are useful even though they arent real numbers.

- $\mathbb{R} \cup \{-\infty, \infty\}$ = all real numbers

$[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$ $(a, \infty) = \{x \in \mathbb{R} : a < x\}$
$(-\infty, b] = \{x \in \mathbb{R} : b \geq x\}$ $(-\infty, b) = \{x \in \mathbb{R} : b > x\}$

$[a, \infty)$ and $(-\infty, b]$ are closed, unbounded intervals
(a, ∞) and $(-\infty, b)$ are open, unbounded intervals

- if $\sup S = +\infty$, S is not bounded above
- if $\inf S = -\infty$, S is not bounded below

if S is bounded above then $\sup S = a$, $a \in \mathbb{R}$
otherwise $\sup S = +\infty$ (Not bounded above)

if S is bounded below then $\inf S = b$, $b \in \mathbb{R}$
otherwis $\inf S = -\infty$ (Not bounded below)

· $\sup(A+B) = \sup A + \sup B$ and $\inf(A+B) = \inf A + \inf B$

Q's:
· Whats the difference between $\mathbb{R} \cup \{-\infty, \infty\}$ and just $\mathbb{R}$?
· What properties of $\mathbb{R}$ hold for $-\infty$ and ∞?
· Why is $[a, \infty)$ closed?

§6 A Development on $\mathbb{R}$
$a = \sup\{r \in \mathbb{Q} : r < a\}$ for each $a \in \mathbb{R}$

$a \leq b$ iff $\{r \in \mathbb{Q} : r < a\} \subseteq \{r \in \mathbb{Q} : r > b\}$ and vice-versa
$a = b$ iff $\{r \in \mathbb{Q} : r < a\} = \{r \in \mathbb{Q} : r < b\}$

subsets α of $\mathbb{Q}$ having the form
$\{r \in \mathbb{Q} : r < \alpha\}$ satisfy these properties
(i) $\alpha \neq \mathbb{Q}$ and α is not empty
(ii) if $r \in \alpha$, $s \in \mathbb{Q}$ and $s < r$, then $s \in \alpha$
(iii) α contains no largest rational

Q's:
· is $\{r \in \mathbb{Q} : r < a\}$ a set of r's or a's?

Chapter 2: sequences
§7 Limits of Sequences
A sequence is a function whose domain is a set of the form $\{n \in \mathbb{Z} : n \geq m\}$; m is usually 1 or 0. So a sequence is a function that has a specific value for each integer $n \geq m$.

Notation

sequence $s \Rightarrow s_n = (s_n)_{n=m}^{\infty}$ or $(s_m, s_{m+1}, s_{m+2}, \ldots)$, if $m=1$, $(s_n)_{n\in\mathbb{N}}$ or $(s_1, s_2, s_3, \ldots)$.

Ex/ $(s_n)_{n\in\mathbb{N}}$ where $s_n = \frac{1}{n^2} = (1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots)$

· the limit of a sequence s_n is a real number that the values s_n are close to for large values of n.

7.1 Definition

A sequence s_n is said to converge to the real number s if

for each $\varepsilon > 0$ there exists a number N s.t. $n > N$ implies $|s_n - s| < \varepsilon$

i.e. for N large enough the difference between the sequence s_n and its limit s is less than ε, which is arbitrary small. So $s_n \to$ =0

Notation: $\lim_{n\to\infty} s_n = s$, or $s_n \to s$

· A sequence that doesn't converge, diverges

> N depends on the value ε.

Ex/ Consider the sequence $s_n = \frac{3n+1}{7n-4}$. We can rewrite s_n as $\frac{3+\frac{1}{n}}{7-\frac{4}{n}}$, and $\frac{1}{n}, \frac{4}{n} \to 0$ so $\lim = \frac{3}{7}$

$\lim s_n = \frac{3}{7}$ means

for each $\varepsilon > 0$ there exists a number N s.t.

$n > N$ implies $\left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < \varepsilon$

as ε varies, N varies

$\Rightarrow n > 0$ implies $\left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < 1$

$n > 4$ implies $\left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < 0.1$

$n > 39$ implies $\left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < 0.01$

$n > 388$ implies $\left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < 0.001$

$n > 387{,}555$ implies $\left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < 0.000001$

- Limits are unique. That is if
  $\lim s_n = s$ and $\lim s_n = t$ then $s = t$
  So s_n cannot be getting arbitarily close to different values for large n

<u>Q's:</u>

- if $\lim s_n = s$ and $\lim t_n = s$ does $s_n = t_n$? if not is there a relation between s_n and t_n?
- Why are we allowed to assume $\lim s_n = s$ as $|s_n - s| < \varepsilon$? even if ε is tiny won't there still be a difference between the two?

<u>§ 9 limit Theorems for Sequences</u>

<u>9.1 thm:</u>

Convergent sequences are bounded.

<u>Pf:</u> Let s_n be a convergent sequence, and let $s = \lim s_n$. Applying 7.1 with $\varepsilon = 1$ we obtain N in $\mathbb{N}$ s.t.

$n > N$ implies $|s_n - s| < 1$

From the triangle inequality we see $n > N$ implies $|s_n| < |s| + 1$

Define $M = \max\{|s|+1, |s_1|, |s_2|, \ldots, |s_N|\}$, then we have $|s_n| \leq M$ for all $n \in \mathbb{N}$, so s_n is a bounded seq.

9.2 thm:

if the sequence (s_n) converges to s, and k in $\mathbb{R}$ then the sequence (ks_n) converges to ks.

that is $\lim(ks_n) = k \cdot \lim s_n$

9.3 thm:

if (s_n) converges to s and (t_n) converges to t, then $(s_n + t_n)$ converges to $s+t$, that is

$$\lim(s_n + t_n) = \lim s_n + \lim t_n$$

9.4 thm:

$$\lim(s_n t_n) = (\lim s_n)(\lim t_n)$$

9.5 lemma:

if (s_n) converges to s, if $s_n \neq 0$ for all n, and if $s \neq 0$ then $(1/s_n)$ converges to $\frac{1}{s}$

9.6 thm:

Suppose s_n converges to s and t_n converges to t. if $s \neq 0$ and $s_n \neq 0$ for all n, then (s_n/t_n) converges to $\frac{t}{s}$

9.7 thm

(a) $\lim_{n\to\infty} \left(\frac{1}{n^p}\right) = 0$ for $p > 0$

(b) $\lim_{n\to\infty} a^n = 0$ if $|a| < 1$

(c) $\lim_{n\to\infty} (n^{1/n}) = 1$

(d) $\lim_{n\to\infty} (a^{1/n}) = 1$ for $a > 0$

9.8 Def:

For a sequence (s_n), we write $\lim s_n = +\infty$ provided for each $M > 0$ there is a number N s.t.

$n > N$ implies $s_n > M$

- we write $\lim s_n = -\infty$ if each $M > 0$ there is an N s.t. $n > N$ implies $s_n < M$

9.9 Thm:

Let s_n and t_n be sequences s.t. $\lim s_n = +\infty$ and $\lim t_n > 0$ [$\lim t_n$ can be finite or ∞] then $\lim s_n t_n = +\infty$

9.10 thm:

For a sequence s_n of positive real numbers, we have $\lim s_n = +\infty$ iff $\lim \left(\frac{1}{s_n}\right) = 0$

Q's:

If $k = \infty$, does $\lim(k \cdot s_n) = \infty$? what if $s_n = 0$?

how do you solve $\lim\left(\frac{t_n}{s_n}\right)$ if $s_n = 0$?

§ 10 Monotone Sequences and Cauchy Sequences

10.1 Def:

A sequence s_n of real numbers is called an increasing sequence if $s_n \leq s_{n+1}$ for all n, and s_n is called a decreasing seq if $s_{n+1} \leq s_n$.

A seq that is increasing or decreasing is called a monotone sequence or monotonic sequence

Ex/ $a_n = 1 - \frac{1}{n^2}$ is increasing

$b_n = \frac{1}{n^2}$ is decreasing

$t_n = (-1)^n$ is not monotonic

10.2 Thm

All bounded monotone sequences converge

10.4 thm

(i) if s_n is an unbounded increasing sequence, then $\lim s_n = +\infty$

(ii) if s_n is an unbounded decreasing sequence, then $\lim s_n = -\infty$

10.6 Def

Let s_n be a sequence in $\mathbb{R}$ we define

$$\lim\sup s_n = \lim_{N \to \infty} \sup\{s_n : n > N\}$$

and

$$\lim\inf s_n = \lim_{N \to \infty} \inf\{s_n : n > N\}$$

s_n doesn't have to be bounded, but if unbounded then $\lim\sup s_n = +\infty$, and $\lim\inf s_n = -\infty$

- $\lim\sup s_n \leq \sup\{s_n : n \in \mathbb{N}\}$
- $\lim\inf s_n \geq \inf\{s_n : n \in \mathbb{N}\}$

10.7 Thm

Let s_n be a seq in $\mathbb{R}$

(i) if $\lim s_n$ is defined then:

$$\lim\inf s_n = \lim s_n = \lim\sup s_n$$

(ii) if $\lim s_n = \lim\sup s_n$, then $\lim s_n$ is defined and $\lim s_n = \lim\inf s_n = \lim\sup s_n$.

10.8 Def:

A sequence s_n of real numbers is called a Cauchy sequence if for each $\varepsilon > 0$ there exists a number N s.t.

$$m, n > N \text{ implies } |s_n - s_m| < \varepsilon$$

10.9 Lemma

Convergent sequences are Cauchy sequences

10.10 Lemma

Cauchy sequences are bounded

10.9 Proof

Suppose $\lim s_n = s$

$$|s_n - s_m| = |s_n - s + s - s_m| \leq |s_n - s| + |s - s_m|$$

let $\varepsilon > 0$, then there exists N s.t.

$n > N$ implies $|s_n - s| < \frac{\varepsilon}{2}$

$m > N$ implies $|s_m - s| < \frac{\varepsilon}{2}$

So $m, n > N$ implies $|s_n - s_m| \leq |s_n - s| + |s - s_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

10.11 Thm

A sequence is a convergent sequence iff it is a Cauchy sequence

Q's:

is $\lim\inf s_n \geq$ or $\leq \inf\{s_n : n \in \mathbb{N}\}$?

is a constant function increasing or decreasing?

Can $\lim\sup s_n \in s_n$? Can $\lim\sup s_n$ be within s_n?

Does a function need to be oscillating to be Cauchy?

§11 Subsequences

11.1 Def

$(s_n)_{n \in \mathbb{N}}$ is a sequence. A subsequence is $(t_k)_{k \in \mathbb{N}}$ where each k there is a positive integer n_k s.t.

$$n_1 < n_2 < \ldots < n_k < n_{k+1} < \ldots$$

and

$t_k = s_{n_k}$, t_k is a selection of some (or all) n_k's taken in order

11.3 Thm

If the sequence (s_n) converges, then every subsequence converges to the same limit.

11.4 Thm

Every sequence (s_n) has a monotonic subsequence

11.5 Bolzano-Weierstrass Thm

Every bounded sequence has a convergent subsequence

11.7 thm

Let s_n be any sequence. There exists a monotonic subsequence whose limit is $\lim\sup s_n$, and there exists a monotonic subsequence whose limit is $\lim\inf s_n$.

11.8 Thm

Let s_n be any sequence in $\mathbb{R}$, and let S denote the set of subsequential limits of s_n

(i) S is nonempty

(ii) $\sup S = \lim\sup s_n$ and $\inf S = \lim\inf s_n$

(iii) $\lim s_n$ exists iff S has exactly one element, namely $\lim s_n$

Q's:

If you wanna prove the limit of a sequence can you use subsequential limits?

Can you have a subsequence with one term?

Can you have a subseq with infinite terms?

§12 lim sup's and lim inf's

Let s_n be any sequence, and let S be the set of subsequential limits on s_n.

$$\lim\sup s_n = \lim_{N\to\infty} \sup\{s_n : n > N\} = \sup S$$

and

$$\lim\inf s_n = \lim_{N\to\infty} \inf\{s_n : n > N\} = \inf S$$

12.1 Thm

if s_n converges to a positive real number s and t_n is any sequence

$$\lim\sup s_n t_n = s \cdot \lim\sup t_n$$

12.2 Thm

Let s_n be any sequence of nonzero real numbers then

$$\liminf \left|\frac{s_{n+1}}{s_n}\right| \le \liminf |s_n|^{1/n} \le \limsup |s_n|^{1/n} \le \limsup \left|\frac{s_{n+1}}{s_n}\right|$$

12.3 Corollary:

if $\lim \left|\frac{s_{n+1}}{s_n}\right|$ exists, and equals L, then $\lim |s_n|^{1/n}$ exists and equals L

Q's:

To use Thm 12.2, do we have to compute $\left|\frac{s_{n+1}}{s_n}\right|$ and $|s_n|^{1/n}$ or is there a trick?

Why is knowing $\left|\frac{s_{n+1}}{s_n}\right|$ and $|s_n|^{1/n}$ useful?

§13 Topology

Def: Metric Space

Let S be a set, suppose d is a function for all pairs (x,y) of elements S satisfying:

D1) $d(x,x) = 0$ for all $x \in S$ and $d(x,y) > 0$ for $\forall x \neq y \in S$

D2) $d(x,y) = d(y,x)$ for $\forall x, y \in S$

D3) $d(x,z) \le d(x,y) + d(y,z)$ for $\forall x, y, z \in S$

The set S with function d is called a Metric space

* just a special case of sets

13.2 Def

A sequence s_n in a metric space (S,d) converges to s in S if $\lim_{n\to\infty} d(s_n, s) = 0$. A sequence s_n in S is a Cauchy sequence if for each $\varepsilon > 0$ there exists an N s.t.

$m,n > N$ implies $d(s_m, s_n) < \varepsilon$

The metric space (S,d) is said to be complete if every Cauchy sequence in S converges to some element in S

13.4 Thm

Euclidean K-Space $\mathbb{R}^K$ is complete

13.5 thm BW

Every bounded sequence in $\mathbb{R}^K$ has a convergent subsequence

13.6 Def

Let (S,d) be a metric space, Let E be a subset of S.
An element $s_0 \in E$ is interior to E if some $r>0$ we have
$$\{s \in S : d(s,s_0) < r\} \subseteq E$$
we write E^0 for the set of points in E that are interior to E. The set E is open in S if every point in E is interior to E, $E = E^0$

13.7

(i) S is open in S

(ii) The empty set $\emptyset$ is open in S

(iii) The union of any collection of open sets is open

(iv) the intersection of finitely many open sets is an open set

13.8 Def:

Let (S,d) be a metric space. A subset E of S is closed if its complement $S \setminus E$ is an open set or E is closed if $E = S \setminus U$ where U is an open set

· boundary points in E := $E^- \setminus E^0$

· The closure of E, $\Rightarrow E^-$ = the intersection of all closed sets containing E

13.9 Prop: Let $E \subseteq (S,d)$

(a) The set E is closed iff $E = E^-$

(b) The set E is closed iff it contains the limit of every convergent seq of points in E

(c) An element is in E^- iff it is in the limit of some seq of points in E

(d) A point is in the boundary of E iff it belongs to the closure of E and $\overline{E}$

Open balls:

open balls $\{x : d(x, x_0) < r\}$ are open sets

Closed balls $\{x : d(x, x_0) \leq r\}$ are closed sets

The boundary of each of these sets is
$\{x : d(x, x_0) = r\}$ in the plane $\mathbb{R}^2$ the sets
$\{(x_1, x_2) : x_1 > 0\}$ and $\{(x_1, x_2) : x_1 > 0$ and $x_2 > 0\}$
are open, if you replace $>$ with $\geq$ they're closed

13.11 Def:

Let (S, d) be a metric space. A family $\mathcal{U}$ of open sets is said to be an open cover for a set E if each point of E belongs to at least one set in $\mathcal{U}$

$$E \subseteq \bigcup \{U : U \in \mathcal{U}\}$$

Compactness:

if every open cover of E has a finite subcover of E

13.12 Heine-Borel Thm

A subset E of $\mathbb{R}^k$ is compact iff it is closed and bounded

Q's:

How do you prove a set is compact?

Can a set be closed and bounded and not be compact?

Why is S open in S?

does $d(s_n, s_n)$ apply to functions?

§14 Series

The infinite series $\sum_{n=m}^{\infty} a_n$ is said to converge if the sequence s_n of partial sums converge to a real number S, in which case we define $\sum_{n=m}^{\infty} a_n = S$ so

$$\sum_{n=m}^{\infty} a_n = S \quad \text{means} \quad \lim s_n = S \quad \text{or} \quad \lim_{n\to\infty}\left(\sum_{k=m}^{n} a_k\right) = S$$

· $\sum a_n$ is said to converge absolutely or to be absolutely convergent if $\sum |a_n|$ converges

Ex 1/
A series of the form $\sum_{n=0}^{\infty} ar^n$ for constants a and r is called a geometric series. They're the easiest series to sum. For $r \neq 1$ the partial sums s_n are given by

$$\sum_{k=0}^{n} ar^k = a\frac{1-r^{n+1}}{1-r}$$

Ex 2/

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p>1$

for $p \leq 1$ we can write $\sum 1/n^p = +\infty$

14.3 Def

We say a series $\sum a_n$ satisfies the Cauchy criterion if the sequence of partial sums is a Cauchy sequence

for each $\varepsilon > 0$ there exists a number N s.t.

$$m, n > N \text{ implies } |s_n - s_m| < \varepsilon$$

or

for each $\varepsilon > 0$ there exists a number N s.t.

$$n \geq m > N \text{ implies } \left|\sum_{k=m}^{n} a_k\right| < \varepsilon$$

14.4 Thm

A series converges iff it satisfies the Cauchy criterion.

14.5 Corollary

if a series Σa_n converges, then $\lim a_n = 0$

14.6 Comparison Test

Let Σa_n be a series where $a_n \geq 0$ for all n

(i) if Σa_n converges and $|b_n| \leq a_n$ for all n then Σb_n converges

(ii) if $\Sigma a_n = +\infty$ and $b_n \geq a_n$ for all n then $\Sigma b_n = +\infty$

14.8 Ratio Test

A series Σa_n of nonzero terms

(i) converges absolutely if $\lim\sup |a_{n+1}/a_n| < 1$

(ii) diverges if $\lim\inf |a_{n+1}/a_n| > 1$

(iii) otherwise $\lim\inf |a_{n+1}/a_n| \leq 1 \leq \lim\sup |a_{n+1}/a_n|$ and the test gives no information

14.9 Root Test

Let Σa_n be a series and let $\alpha = \lim\sup |a_n|^{1/n}$ then Σa_n

(i) converges absolutely if $\alpha < 1$

(ii) diverges if $\alpha > 1$

(iii) otherwise $\alpha = 1$ and the test gives no inf

Alternating Series Test:

thm: Let $a_1 \geq a_2 \geq a_3 \geq \ldots$ be a monotone decreasing series $a_n \geq 0$, A assuming $\lim a_n = 0$. Then

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 \ldots$$

converges

Integral Test: Replace Σ_n by $\int x(\ldots)\,dx$

- works for positive sequences

Q's:

whats a partial sum?

For the comparison test do the series have to be similar?

if $\lim a_n = a$, $a \in \mathbb{R}$, does $\sum a_n$ converge?

Chapter 3 Continuity

17.1 Def:

let f be a real-valued function. The function f is continuous at x_0 in dom(f), if for every sequence (x_n) in dom(f) converging to x_0 we have

$$\lim_n f(x_n) = f(x_0)$$

17.2 Thm

Let f be a real valued function whose domain is a subset in $\mathbb{R}$ the f is continuous at x_0 in dom(f) iff

for each $\varepsilon > 0$ there exists $\delta > 0$ s.t.

$x \in dom(f)$ and $|x - x_0| < \delta$ imply $|f(x) - f(x_0)| < \varepsilon$

17.4 Thm

Let f and g be real-valued functions that are continuous at x_0 in $\mathbb{R}$ then

(i) $f+g$ is continuous at x_0

(ii) fg is continuous at x_0

(iii) f/g is continuous at x_0 if $g(x_0) \neq 0$

17.5 Thm

If f is continuous at x_0 and g is continuous at $f(x_0)$ the composite function $g \circ f$ is continuous at x_0

Q's: Pointwise continuity just means continuous at a point?

§18 Properties of Continuous Functions

18.1 Thm

if f is Continuous and bounded it has max and min

$\Rightarrow f(x_0) \leq f(x) \leq f(y_0)$

18.2 Intermediate Value theorem

if f is continuous real-valued function on an interval I, then f has the intermediate value property on I: whenever $a, b \in I$, $a < b$ and y lies between $f(a)$ and $f(b)$, $f(a) < y < f(b)$ or $f(b) < y < f(a)$ there exists at least one x in (a,b) s.t. $f(x) = y$

18.4 Thm

if $f(I)$ on interval J is continuous and strictly increasing then f^{-1} is a continuous and strictly increasing function on J

Q's:

if $f(I)$ is continuous and not strictly increasing, is f^{-1} still continuous?

§19 Uniform Continuity

A function is uniformly continuous if for each $x_0 \in S$ and $\varepsilon > 0$ so that $x \in dom(f)$ and $|x - x_0| < \delta$ imply $|f(x) - f(x_0)| < \varepsilon$

and δ doesn't depend on x_0

19.2 Thm

If f is continuous on a closed interval $[a,b]$ then f is uniformly continuous on $[a,b]$

Q's: Does δ depend on ε, for uniform continuity?

Rudin

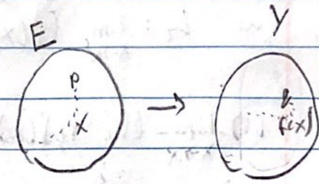
Chapter 4

Limits of functions

4.1 Def

Let X and Y be metric spaces, suppose $E \subset X$, f maps E into Y, and p is a limit point of E.

We write $f(x) \to q$ as $x \to p$ or



(1) $\lim_{x \to p} f(x) = q$

$\exists q \in Y$, for every $\varepsilon > 0$ there exists $\delta > 0$ s.t.

(2) $d_Y(f(x), q) < \varepsilon$

for all points $x \in E$ for which

(3) $0 < d_X(x, p) < \delta$

4.2 Thm

Let X, Y, E, f and p be as above then

$$\lim_{x \to p} f(x) = q \quad \text{iff} \quad \lim_{n \to \infty} f(p_n) = q$$

for every sequence p_n in E s.t. $p_n \neq p$, $\lim_{n \to \infty} p_n = p$

Q's:

Why isn't it $\lim d_Y(f(x), q) < \varepsilon$?

Can you take a lim of a distance?

Does p have to an element of E? as its a limit point

Continuous Functions

4.5 Def: Suppose X and Y are metric spaces, $E \subset X$, $p \in E$ and f maps E into Y. f is continuous at p if for every $\varepsilon > 0$ there exist an $\delta > 0$ s.t. $d_Y(f(x), f(p)) < \varepsilon$ with $d_X(f(x), f(p)) < \varepsilon$

if f is continuous at every point on E, then f is continuous on E. f has to be defined at p to be continuous at p.

20.4 thm

Let f_1 and f_2 be functions for which the limits $L_1 = \lim_{x \to a^S} f_1(x)$ and $L_2 = \lim_{x \to a^S} f_2(x)$ exist and are finite. then

(i) $\lim_{x \to a^S} (f_1 + f_2)(x) = L_1 + L_2$

(ii) $\lim_{x \to a^S} (f_1 f_2)(x) = L_1 L_2$

(iii) $\lim_{x \to a^S} (f_1 / f_2)(x) = L_1 / L_2$, $L_2 \neq 0$ and $f_2 \neq 0$

20.6 thm

Let f be a function on $S \subset \mathbb{R}$

$a = \lim$ of some sequence in S

$L \in \mathbb{R}$, $\lim_{x \to a^S} f(x) = L$ iff

for each $\varepsilon > 0$ there exists $\delta > 0$ s.t. $x \in S$ and $|x - a| < \delta$ imply $|f(x) - L| < \varepsilon$

20.10 thm

let f be a function defined on $J \setminus \{a\}$ for some open interval J containing a. then $\lim_{x \to a^S} f(x)$ exists iff the limits $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ both exist and are equal, so all three limits are the same.

Q's:

how is $\lim_{x \to a^S}$ different from $\lim_{x \to a}$?

Why do we define $J \setminus \{a\}$, the exclusion on a?

§21 More on metric spaces: continuity

21.1 Def

Consider metric spaces (S,d) and (S^*,d^*). A function $f: S \to S^*$ is continuous at s_0 in S if

for each $\epsilon > 0$ there exists $\delta > 0$ s.t.
$d(s,s_0) < \delta$ implies $d^*(f(s), f(s_0)) < \epsilon$

a function is uniformly continuous if
for each $\epsilon > 0$ there exists $\delta > 0$ s.t.
$s,t \in E$ and $d(s,t) < \delta$ imply $d^*(f(s), f(t)) < \epsilon$

21.3 thm

Consider metric spaces (S,d) and (S^*,d^*). A function $f: S \to S^*$ is continuous on S iff

$f^{-1}(U)$ is open

Recall: $f^{-1}(U) = \{s \in S : f(s) \in U\}$

21.4 thm

Consider the metric spaces (S,d), (S^*,d^*) and a continuous function $f: S \to S^*$. Let E be a compact subset on S. Then
(i) $f(E)$ is a compact subset on S^* and
(ii) f is uniformly continuous on E

continuous on compact set $\Rightarrow$ uniform continuity

21.5 Cor

Let f be a continuous real-valued function on a metric space (S,d). If E is a compact subset of S, then
(i) f is bounded on E
(ii) f assumes its maximum and minimum on E

A subset D of S is dense in S if every nonempty open set U intersects D, i.e. $D \cap U \neq \emptyset$

Ex/ $\mathbb{Q}$ is dense in $\mathbb{R}$, b/c every nonempty open interval in $\mathbb{R}$ contains rations

Q's:

What the minimum number of elements needs in an interval?

§22 More on Metric Spaces Connectedness

A topological space

a set X together with a collection of subsets of X, called open subsets s.t.

- X, $\emptyset$ are open
- $\bigcup_{\alpha \in A} U_\alpha$ is open   arbitrary union of the subsets is open
- $\bigcap_{i=1}^{n} U_i$ is open   finite intersection

- A subset of a topological space has induced topology $U \subset S$ is open iff $\exists \tilde{U} \subset X$, open in X, s.t. $U = \tilde{U} \cap S$

Cor: (a) if $S \subset X$ is open in X, then $U \subset S$ is open in S iff U is open in X

(b) if $S \subset X$ is closed in X, then $E \subset S$ is closed in S iff E is closed in X

Recall: A top space X is connected if the only subsets of X that is both open and closed are X and $\emptyset$

- X is <u>not connected</u> iff $X = U \amalg V$ (disjoint union)

U, V are non-empty subsets of X.