Q's:

how can a set be both open and closed?

From Lecture Notes

Discontinuity for Real-Valued Functions

Recall $f: X \to Y$ is discontinuous at $x \in X$ iff x is a limit point of X, and $\lim_{q \to x} f(q)$ either doesn't exist or does not equal to $f(x)$

Def:

Let $f: (a,b) \to \mathbb{R}$ (not necessarily continuous)

· $\forall x \in [a,b)$, we say $f(x+) = q$, if for all seq (t_n) in (x,b) that converge to x, we have limit $\lim_n f(t_n) = q$

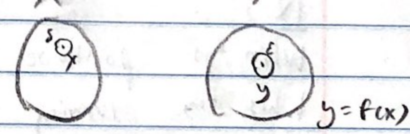
· $\forall x \in (a,b]$, we say $f(x-) = q$ if for all seq (t_n) in (a,x), converging to x we have $\lim_{n \to \infty} f(t_n) = q$

Def: $f: (a,b) \to \mathbb{R}$ $x_0 \in (a,b)$ suppose f is not continuous at x_0

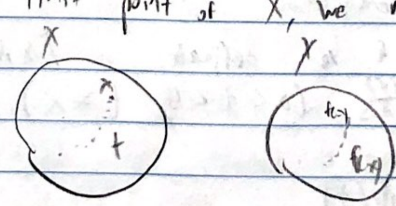
· We say f has a simple discontinuity at x_0, if both $f(x+)$ and $f(x-)$ exist, and either $f(x_0+) \neq f(x)$ or $f(x_0-) \neq f(x)$

Continuous Maps: Let $f: (X, d_X) \to (Y, d_Y)$ be a map between metric spaces. The def for continuity

Def 1: $\forall x \in X$, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $f(B_\delta(x)) \subset B_\varepsilon(f(x))$



Def 2: $\forall x \in X$, a limit point of X, we require $\lim_{t \to x} f(t) = f(x)$



Def 3: For any open subset $V \subset Y$, $f^{-1}(V)$ is an open subset in X

Property:

(1) if $f: X \to Y$ continuous, $K \subset X$ compact then $f(K)$ is compact

(2) if $f: X \to Y$ continuous, $K \subset X$ is connected, $f(K)$ is connected

Uniform Continuity:

$f: (X, d_X) \to (Y, d_Y)$ is uniformly continuous if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $\forall x, y \in X$, if $d(x,y) < \delta$ then $d(f(x), f(y)) < \varepsilon$

Sequences of Functions

Let $f_n: X \to Y$ be a sequence of functions between metric spaces. Let $f: X \to Y$ be a function. We say $f_n \to f$ pointwise if $\forall x \in X$.

$$\lim_{n \to \infty} f_n(x) = f(x) \iff \lim_{n \to \infty} |f_n(x) - f(x)| = 0$$

We say, $f_n \to f$ uniformly if

$$\lim_{n \to \infty} \underbrace{\sup_{x \in X} |f_n(x) - f(x)|}_{d_\infty(f_n, f)} = 0$$

Q's:

When is pointwise continuity useful?

What are running bumps?

Chapter 5: Differentiation

5.1 Def: Let f be defined on $[a,b]$ for any $x \in [a,b]$

$$\phi(t) = \frac{f(t) - f(x)}{t - x} \quad (a < t < b, \ t \neq x)$$

$$f'(x) = \lim_{t \to x} \phi(t)$$

5.2 thm:

Suppose f and g are defined on $[a,b]$ and are differentiable at a point $x \in [a,b]$, then

(a) $(f+g)'(x) = f'(x) + g'(x)$
(b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
(c) $(f/g)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

5.5 thm:

Suppose f is continuous on $[a,b]$, $f'(x)$ exists at some point $x \in [a,b]$, g is defined on an interval I, which contains the range of f, and g is differentiable at the point $f(x)$; if

$h(t) = g(f(t)) \quad (a \le t \le b)$

then h is differentiable at x, and

$h'(x) = g'(f(x)) f'(x)$

5.7 Def:

let f be a real function defined on a metric space X, we say that f has a local maximum at a point $p \in X$ if there exists $\delta > 0$ s.t. $f(q) \le f(p)$ for all $q \in X$ with $d(p,q) < \delta$

5.9 thm: if f and g are continuous real functions on $[a,b]$ which are differentiable in (a,b), then there is a point $x \in (a,b)$ at which

$[f(b) - f(a)] g'(x) = [g(b) - g(a)] f'(x)$

5.10 thm

if f is a real continuous function on $[a,b]$ which is differentiable in (a,b), then there is a point $x \in (a,b)$ at which
$f(b) - f(a) = (b-a) f'(x)$

Q: What is the second kind of discontinuity

5.11 thm: Suppose f is differentiable in (a,b)

(a) if $f'(x) \geq 0$ for all $x \in (a,b)$ then f is monotonically increasing

(b) if $f'(x) = 0$ for all $x \in (a,b)$ then f is a constant

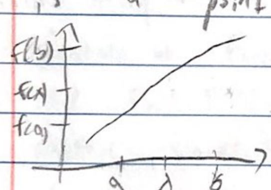
(c) if $f'(x) \leq 0$ for all $x \in (a,b)$ then f is monotonically decreasing

Q's:

What do thm's 5.9 and 5.10 tell us?

The continuity of Derivatives:

5.12 Thm: Suppose f is a real differentiable function on $[a,b]$ and suppose $f'(a) < \lambda < f'(b)$. Then there is a point $x \in (a,b)$ s.t. $f'(x) = \lambda$



similar result holds if $f'(a) > f'(b)$

Cor: if f is differentiable on $[a,b]$ then f' cannot have any simple discontinuities, but f' may have discontinuities of the second kind.

L'Hospital's Rule

5.13 thm: Suppose f and g are real and differentiable in (a,b), and $g'(x) \neq 0$ for all $x \in (a,b)$, where $-\infty < a < b < \infty$. Suppose

$$\frac{f'(x)}{g'(x)} \to A \quad \text{as} \quad x \to a$$

if $f(x) \to 0$ as $x \to a$ or if $g(x) \to +\infty$ as $x \to a$

then

$$\frac{f(x)}{g(x)} \to A \quad \text{as} \quad x \to a$$

or $$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Q's:

What are discontinuities of the second kind?

For L'Hopital's rule what if $f'(x) \to \infty$ or 0?

Derivatives of Higher Derivative

Taylor's Theorem

Suppose f is a real function on $[a,b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a,b]$, $f^{(n)}(t)$ exists for every $t \in (a,b)$. Let α, β be distinct points of $[a,b]$, and define

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k$$

then there exists a point x between α and β s.t.

$$(24)\ f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n$$

for $n=1$, its just the mean value theorem. The theorem shows that f can be approximated by a polynomial of degree $n-1$

Q's:

is taylor's thm an estimate for higher derivatives?

taylor's thm only tell's you higher derivatives at certain points?

What does def 24 tell us?

Chapter 6: The Riemann-Stieltjes Integral

6.1 Def: Let $[a,b]$ be a given interval. By a partition P of $[a,b]$, we mean a finite set of points $x_0, x_1, \ldots, x_n$ where

$$a = x_0 \le x_1 \le \ldots \le x_{n-1} \le x_n = b$$

we write

$$\Delta x_i = x_i - x_{i-1} \quad (i = 1, \ldots, n)$$

Now suppose f is a bounded real function defined on $[a,b]$, corresponding to each partition P of $[a,b]$ we put.

$$\left.\begin{array}{l} M_i = \sup f(x) \quad (x_{i-1} \leq x \leq x_i) \\ m_i = \inf f(x) \quad (x_{i-1} \leq x \leq x_i) \end{array}\right\} \text{each partition has an } M_i \text{ are } m_i$$

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$$
$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$$

and,

$$\overline{\int_a^b} f\,dx = \inf U(P,f) \quad \text{upper integral}$$
$$\underline{\int_a^b} f\,dx = \sup L(P,f) \quad \text{lower integral}$$

if $\inf U(P,f) = \sup L(P,f)$ then f is Riemann integral or $f(x) \in \mathcal{R}$

if so we write $\int_a^b f\,dx$

This is the Riemann integral of f over $[a,b]$.

Since f is bounded, there exist two numbers, m and M s.t.

$$m \leq f(x) \leq M \quad (a \leq x \leq b)$$

Hence for every P,

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$$

So, the upper and lower integrals are defined for every bounded function f.

6.2 Def:

Let α be a monotonically increasing function on $[a,b]$, since $\alpha(a)$ and $\alpha(b)$ are finite, it follows that α is bounded on $[a,b]$. Corresponding to each partition P of $[a,b]$ we write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

its clear that $\Delta\alpha_i \geq 0$ for any real function f which is bounded on $[a,b]$ we put

$$U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$$
$$L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$

we define

$$\overline{\int_a^b} f\,d\alpha = \inf U(P,f,\alpha) \qquad \text{if } \overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$$
$$\underline{\int_a^b} f\,d\alpha = \sup L(P,f,\alpha) \qquad \Rightarrow \int_a^b f(x)\,d\alpha(x) \quad \Big\} \text{ Stieltjes integral}$$

if exist we say f is integrable with respect to α or $f(x) \in \mathcal{R}(\alpha)$

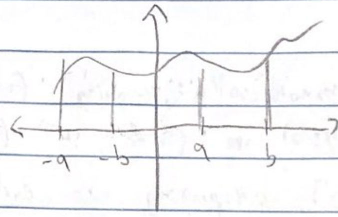
taking $\alpha(x) = x$ shows the Riemann integral is a special case of Stieltjes integral

6.3 Def: we say that the partition P^* is a refinement of P if $P^* \supset P$ (if every point of P is a point in P^*)
Given two partitions P_1 and P_2, we say that P^* is their common refinement if $P^* = P_1 \cup P_2$

6.4 Thm: if P^* is a refinement of P then,
$$L(P,f,\alpha) \leq L(P^*,f,\alpha) \quad \text{and}$$
$$U(P^*,f,\alpha) \leq U(P,f,\alpha)$$

6.5 thm

$\int_{-a}^{b} f\,d\alpha \le \int_{-b}^{a} f\,d\alpha$



6.6 thm

$f \in \mathcal{R}(\alpha)$ on $[a,b]$ iff for every $\varepsilon > 0$ there exists a partition P s.t.

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$$

True for any refinement P^* on $[a,b]$

$\Rightarrow$ if true for $P: \{x_0, \dots, x_n\}$ and if s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$ then

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i < \varepsilon$$

then

$$\left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f\,d\alpha \right| < \varepsilon$$

6.8 thm

if f is continuous on $[a,b]$ then f is integrable on $[a,b]$

6.9 thm

if f is monotonic and α is continuous on $[a,b]$ then $f \in \mathcal{R}(\alpha)$ (and assuming α is monotonic)

6.10 thm: if f is bounded on $[a,b]$, and has only finitely many points of discontinuity on $[a,b]$, and α is continuous at every point at which f is discontinuous then $f \in \mathcal{R}(\alpha)$

6.1 thm

Suppose $f \in \mathcal{R}(\alpha)$ on $[a,b]$, $m \leq f \leq M$, ϕ is continuous on $[m,M]$, and $h(x) = \phi(f(x))$ on $[a,b]$, then $h \in \mathcal{R}(\alpha)$ on $[a,b]$

Q's

What is K?

What is a weight function?

what's the difference between $L(P,f)$ and $L(P,f,\alpha)$?

if is bounded is f integrable?

if f has infinitely many discontinuities on an infinite interval is it not integrable?

Integration and Differentiation

Differentiation is the inverse of Integration

6.20 thm: Let $f \in \mathcal{R}(\alpha)$ on $[a,b]$ for $a \leq x \leq b$ put

$$F(x) = \int_a^x f(t)\, dt$$

Then F is continuous on $[a,b]$, and F is continuous at point x_0 and differentiable at x_0

6.21 The Fundamental Theorem of Calculus

if $f \in \mathcal{R}$ on $[a,b]$ and if there is a differentiable function F on $[a,b]$ s.t. $F' = f$ then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

i.e. integral of a derivative, is the function at b minus the function at a

6.22 Thm integration by parts

Suppose F and G are differentiable functions on $[a,b]$

$F' = f \in \mathcal{R}$ and $G' = g \in \mathcal{R}$ then

$$\int_a^b F(x) g(x)\, dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x) G(x)\, dy$$

Q's:

is there a way to show differentiation is the inverse of integration?

Properties of the Integral

6.12 thm

(a) if $f_1 \in \mathcal{R}(\alpha)$ and $f_2 \in \mathcal{R}(\alpha)$ on $[a,b]$ then

$f_1 + f_2 \in \mathcal{R}(\alpha)$,

$cf \in \mathcal{R}(\alpha)$ for every constant c and

$$\int_a^b (f_1 + f_2)\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha$$

$$\int_a^b cf\, d\alpha = c\int_a^b f\, d\alpha$$

(b) if $f_1(x) \le f_2(x)$ on $[a,b]$ then

$$\int_a^b f_1\, d\alpha \le \int_a^b f_2\, d\alpha$$

(c) if $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and if $a < c < b$, then $f \in \mathcal{R}(\alpha)$ on $[a,c]$ and on $[c,b]$ then

$$\int_a^c f\, d\alpha + \int_c^b f\, d\alpha = \int_a^b f\, d\alpha$$

(d) if f is bounded above then

$$\left|\int_a^b f\, d\alpha\right| \le M[\alpha(b) - \alpha(a)]$$

e) if $f \in \mathcal{R}(\alpha_1)$ and $f \in \mathcal{R}(\alpha_2)$ then $f \in \mathcal{R}(\alpha_1 + \alpha_2)$ and

$$\int_a^b f\, d(\alpha_1 + \alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2$$

6.13 thm

if $f \in \mathcal{R}(\alpha)$ and $g \in \mathcal{R}(\alpha)$ on $[a,b]$ then

(a) $fg \in \mathcal{R}(\alpha)$

(b) $|f| \in \mathcal{R}(\alpha)$ and $\left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f|\,d\alpha$

Q's:

What is the function $d\alpha$?

Why are we allowed to change the variable of integration?

is α a function?