Number systems

Prove $x \in \mathbb{Q}$: if $r = \frac{c}{d} \in \mathbb{Q}$ is a rational #

then r satisfies

$$c_n \cdot x^n + c_{n-1} x^{n-1} + \dots + c_0 = 0$$

w/ $c_i \in \mathbb{Z}$, $c_n \neq 0$ $\begin{cases} d \mid c_n \\ c \mid c_0 \end{cases}$ coprime

suppose $\sqrt{2}$ is rational $\pm 1, 2$

$\sqrt{2}$ is root of $x^2 - 2 = 0 \Rightarrow \sqrt{2}$ is

an integer → impossible, $\sqrt{2}$

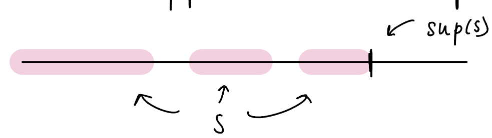
not rational #

→ only rational sol of this eq must be

an int that divides c_0

$|a+b| \leq |a| + |b|$

completness axiom: every nonempty subset of $\mathbb{R}$ that is bounded above has a least upper bound (sup S exists)



- max(S) may not exist : open intervals
  ↳ exists if finite

} upper/ lower bound may not always exist

- $\sup(S) = \min\{\alpha \mid \alpha$ is upper bound$\}$
- $\inf(S) = \max\{\alpha \mid \alpha$ is lower bound$\}$
  ↳ $\inf(S) = -\infty \iff$ not bounded below

if $\max(S) = \sup(S)$, $\inf(S) = \min(S)$

and if S is connected

$\Rightarrow$ S is a closed (bound) int

Sequences & Limits

: limit $\alpha \in \mathbb{R}$ if $\forall \varepsilon > 0$ $\exists N > 0$ s.t. $\forall n > N$ $|a_n - \alpha| < \varepsilon$

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

Ex $\lim_{n \to \infty} \frac{1}{n^2} = 0$ $\quad |a_n - 0| < \varepsilon \rightarrow$

$$\frac{1}{n^2} < \varepsilon \rightarrow \frac{1}{\sqrt{\varepsilon}} < n \Rightarrow N = \frac{1}{\sqrt{\varepsilon}} \qquad \#$$

Thrm : All convergent sequences are bounded

→ if $\lim a_n = \alpha \Rightarrow \left(\frac{1}{a_n}\right)$ is bounded sequence

- $\lim \frac{1}{n^p} = 0 \quad p > 0$

→ $\lim a_n = +\infty$: if $\forall M > 0$, $\exists N > 0$ s.t. $a_n > M$ $\forall n > N$

Monotone / Cauchy

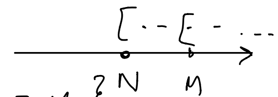
$\Rightarrow$ inc seq : $a_{n+1} \geq a_n$, dec: $a_{n+1} \leq a_n$

Thrm: If (a_n) is increasing and bound $\Rightarrow$ a_n is convergent

↳ all bounded monotone seq are convergent

Def: lim sup / lim inf

- $S_N := \sup\{a_n \mid n \geq N\}$ $\to$ sup of tail part of a_n



- if $N < M$, $S_N \geq S_M$ $\because \{a_n \mid n \geq N\} \supset \{a_n \mid n \geq M\}$
- S_N is a decreasing sequence $\to$ monotone convergence, S_N has lim

$$\limsup a_n := \lim_{N \to \infty} S_N = \lim_{N \to \infty} \left(\sup_{n > N} (a_n) \right)$$

Cauchy: a_n cauchy if $\forall \varepsilon > 0, \exists N > 0$ s.t. $\forall n, m > N$

$$|a_n - a_m| < \varepsilon$$

Thrm Cauchy $\Longleftrightarrow$ convergent

Pf: if a_n converges to α

$$|a_{n_1} - a_{n_2}| < \varepsilon \to |(a_{n_1} - \alpha) - (a_{n_2} - \alpha)| \leq \underbrace{|a_{n_1} - \alpha|}_{\leq \frac{\varepsilon}{2}} + \underbrace{|a_{n_2} - \alpha|}_{\leq \frac{\varepsilon}{2}} = \frac{\varepsilon}{2}$$

since convergent

Thrm: converges iff $\limsup(a_n) = \liminf(a_n)$ $\Big\} = \lim a_n$

$\to$ bounded seq $\forall \varepsilon > 0, \exists N > 0$ s.t. $\forall n > N$ $a_n < \limsup a_n + \varepsilon$

Recursive Seq

Ex $S_1 = 5$, $S_n = \dfrac{S_{n-1} + 5}{S_{n-1}}$

$\begin{cases} A_n: & S_n \leq S_{n-1} \\ B_n: & S_n \geq \sqrt{5} \end{cases}$ | prove by induction $\to$ bounded below ✓

$$\lim S_{n+1} = \lim \frac{S_n^2 + 5}{2 \cdot S_n} = \frac{\lim(S_n^2 + 5)}{\lim(2 \cdot S_n)}$$

$\Rightarrow \alpha = \lim S_n$ exists

$$\Rightarrow \alpha = \frac{\alpha^2 + 5}{2 \cdot \alpha} \quad \cdots \to \alpha^2 = 5 = +\sqrt{5} \quad \text{since } \alpha > 0$$

Subsequence

S_n is a sequence, n_k be strictly inc seq $\in \mathbb{N}$

$t_k := S_{n_k}$ $\forall k = 1, 2 \ldots$ $\Rightarrow$ t_k subseq $(S_{n_k})_k$

→ can generate countable subsequences

Thrm: s_n, any seq and $t \in \mathbb{R}$ then s_n has subseq converge to t iff $\forall \varepsilon > 0$, the set $A_\varepsilon = \{n \in \mathbb{N} \mid |s_n - t| < \varepsilon\}$ is infinite

↳ infinitely many terms in (s_n) inside $(t-\varepsilon, t+\varepsilon)$

Thrm: Every seq has a monotone subseq

- ∞ dominat terms $\forall m > n$, $s_n > s_m$ → strictly dec
- finite dominant terms − monotone increasing seq

Thrm: every bounded seq has convergent subseq

prove bounded seq is cauchy → convergent

Def: t is a subseq limit of s_n if $\exists$ a subsequence of s_n whose limit is t

- lim sup s_n and lim inf s_n are subsequence limits of any sequence s_n

→ closed subset if $\forall$ convergent sequences in S, the limit also belongs to S

Thrm: any bounded sequence s_n, S = set of subseq limits → S closed

Thrm: $s_n > 0$

$$\liminf \left(\frac{s_{n+1}}{s_n}\right) \leq \liminf (s)^{\frac{1}{n}} \leq \limsup (s_n)^{\frac{1}{n}} \leq \limsup \left(\frac{s_{n+1}}{s_n}\right)$$

Metric Space

$(S \times d)$

↳ set

↳ distance mapping function

$d(x,y) \geq 0$, $d(x,y) = 0 \Rightarrow x = y$

$d(x,y) = d(y,x)$

$d(x,y) + d(y,z) \geq d(x,z)$

if $A \subset S \Rightarrow (A, d)$ is metric space

→ cauchy sequence in S: $\forall \varepsilon > 0$, $\exists N > 0$ s.t. $\forall n, m > N$ $d(s_n, s_m) < \varepsilon$

↳ $(s_n)_n$

→ S_n converges to s if $\forall \varepsilon>0, \exists N>0$ s.t. $\forall n>N$ $d(s_n, s)<\varepsilon$

Def: metric space is complete if every cauchy sequence has a limit in S

ie: $\mathbb{R}^n$

Thrm: Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence

Topology : a collection open of subsets on a set

- $S, \emptyset$ are open
- union of a collection of open subsets is open
- intersection of a finite collection of open subsets is open

$B_r(p) = \{x \in S \mid d(p,x) < r\}$ → open balls $\forall r>0, p \in S$

$\Rightarrow$ $U \subset S$ is open if $\forall p \in U$

$\exists r>0$ s.t. $B_r(p) \subset U$ (can make open ball in U)

$$U = \bigcup_{p \in U} B_{r(p)}(p)$$

* punctured open ball does not include center point p

Def: $E \subset S$ closed iff $E^c = S \setminus E$ is open

$\forall x \notin E, \exists \delta>0$ s.t. $B_\delta(x) \cap E = \emptyset$

→ arbitrary intersection of closed subsets is closed

finite union of closed subsets is closed ($S, \emptyset$ closed)

Def (closure) $\bar{E} = \bigcap \{F \mid F \subset S$ closed set, $F \supset E\}$

E° interior, $\partial E = \bar{E} \setminus E^\circ$

(limit point) : $p \in S$ is limit point of E if $\forall \varepsilon>0, \exists q \in E, q \neq p$ s.t. $d(p,q)<\varepsilon$

↳ E' = set of limit points of E

$\bar{E} = E \cup E'$

Prop $E \subset S$ closed $\Longleftrightarrow$ $\forall$ convergent sequences $x_n \in S$, $\lim x_n = x \in E$

i.e. $E = (0, 1]$ not closed, $\frac{1}{n} \in E$, $0 \notin E$

Def: (open cover) a collection of open sets $\{G_\alpha\}_{\alpha \in A}$ s.t. $E \subset \bigcup_\alpha G_\alpha$

(compact) $K \subset S$ compact if for any open cover of K, can find a finite subcover (finite subset of $\{G_\alpha\}$)

- finite subset, $\{1, \frac{1}{2}, \dots\} \cup \{0\}$: comp
- $\{1, \frac{1}{2}, \frac{1}{3} \dots\}$, $\mathbb{R}$: not comp

Thrm: $K \subset \mathbb{R}^n$

K compact $\iff$ K is closed and bounded

→ closed subset in compact set is compact

→ n-cells in $\mathbb{R}^n$ compact $[a_1, b_1] \times \dots [a_n, b_n]$

Series

$\sum_{n=1}^{\infty} a_n$ → converges iff the sequence S_n converges

$S_n = \sum_{j=1}^{n} a_n$

→ cauchy if $|\sum_{j=n}^{m} a_n| < \varepsilon$

→ cauchy of $\sum_n a_n$ $\iff$ Cauchy of (S_n) $\iff$ Convergence of $\sum a_n$

Cor if $\sum a_n$ converges then $\lim a_n = 0$

$\sum a \cdot r^n = a \cdot \frac{1}{1-r}$, $|r| < 1$

Comparison test $\sum_n a_n$ converges, $|b_n| < a_n$, then $\sum b_n$ converges

→ absolute convergence if $\sum |b_n|$ converges (not always)

- $\alpha = \limsup |a_n|^{\frac{1}{n}}$

  $\alpha > 1$ $\sum a_n$ diverges

  $\alpha < 1$ $\sum a_n$ converges absolutely

  $\alpha = 1$ no info

  Root test $\left(\frac{n^4}{2^n}\right)^{\frac{1}{n}}$, $\frac{n^2}{3^n}$

Ratio test: $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$, $\sum a_n$ converges absolutely

$\left(\frac{2^n}{n!}\right)$ $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$, $\sum a_n$ diverges

Alternating Series $\sum (-1)^{n+1} a_n$

Thrm: $\lim a_n = 0$, $a_1 \geq a_2 \ldots$ then $\sum (-1)^{n+1} a_n$ converges

Integral test $\sum \frac{1}{n^p} < \infty$ if $p > 1$

$\sum \frac{1}{n^p}$ converges iff $\int_1^\infty \frac{1}{x^p} dx < \infty$

Functions

$A \to B$

- injective if $\forall x, y \in A$ and $x \neq y$, then $f(x) \neq f(y)$
- surjective if $f(A) = B$, $\forall \beta \in B$, $\exists \alpha \in A$ s.t. $f(\alpha) = \beta$
- $f^{-1}(E) = \{\alpha \in A \mid f(\alpha) \in E\}$

Thrm: $A' \subset A$, $B' \subset B$, $f: A \to B$ then $f(A') \subset B' \iff A' \subset f^{-1}(B')$

Def: $f: X \to Y$ continuous at $p \in X$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$\forall x \in X$ with $d_x(x, p) < \delta \Rightarrow d_y(f(x), f(p)) < \varepsilon$

$f(B_\delta(p)) \subset B_\varepsilon(f(p))$

Thrm: continuous iff $\forall V \subset Y$ open $f^{-1}(V)$ open

Def (limit of func) $E \subset X$ subset and $f: E \to Y$. Suppose p is a limit point of E, $\lim_{x \to p} f(x) = q$ if there is a point $q \in Y$ s.t. $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $f(B_\delta^*(p) \cap E) \subset B_\varepsilon(q)$

i.e. $0 < d_x(x, p) < \delta \Rightarrow d_y(f(x), q) < \varepsilon$ $\forall x \in E$

Thrm: $\lim_{x \to p} f(x) = q$ iff $\forall$ convergent seq $p_n \to p$ w/ $p_n \in E$

$\lim_{n \to \infty} f(p_n) = q$

Thrm: $f: X \to Y$ f is cont iff $\forall p \in X'$ a limit pt of X

$f(p) = \lim_{x \to p} f(x)$ | $f(\lim_{x \to p} x) = \lim_{x \to p} f(x)$

f, g cont: $f+g$, $f \cdot g$, f/g cont $(g \circ f)x = g(f(x))$ cont.

Thrm: $f: X \to \mathbb{R}^n$ $\quad f(x) = (f_1(x), \ldots f_n(x))$

f is cont $\iff f_i : X \to \mathbb{R}$ are cont

T/F: $f: X \to Y$ cont

$\forall U \subset X$ open, $f(U)$ open (F)

$\forall E \subset Y$ closed, $f^{-1}(E)$ closed (T)

Compact subset properties

- K compact $\Rightarrow$ K bounded, closed
- $E \subset K$ (compact) is closed $\to$ E is compact

Induced topology if X is topological space, $S \subset X$

can equip S w/ induced topology $E \subset S$, E is open in S

iff $\exists$ an open subset $\tilde{E} \subset X$ s.t. $E = S \cap \tilde{E}$

$\to$ preserves distance $\Rightarrow$ continuous

$\to$ compactness is intrinsic, does need to be relative

3 def of continuous maps

f cont $\iff \forall p \in X, \forall \varepsilon > 0, \quad \exists \delta > 0$ s.t.

$\forall x \in X$ with $d_X(x,p) < \delta$, $d_Y(f(p), f(x)) < \varepsilon$

$f(B^X_\delta(p)) \subset B^Y_\varepsilon(f(p))$

$\iff \forall V \subset Y$ open, $f^{-1}(V)$ is open in X

$\iff \forall$ convergent seq $x_n \to x$ in X $\quad f(x_n) \to f(x)$ in Y

Thrm: $f: X \to Y$ cont, $E \subset X$ compact, $f(E) \subset Y$ is compact

$\to$ can find finite subcover

$\to$ compactness $\iff$ sequential compactness

Thrm: If $f: X \to \mathbb{R}$ cont, $E \subset X$ compact then $\exists p, q \in E$

s.t. $f(p) = \sup(f(E))$
$f(q) = \inf(f(E))$ $\Big\}$ $f(E)$ closed & bounded $\in \mathbb{R}$

ex : $K = (0, 1]$ as a subset of $\mathbb{R}$ not closed → not compact
↳ does not admit finite subcover

→ compact subset of $(0, \infty)$ are those compt subset of $\mathbb{R}$ that happens to be in $(0, \infty)$

→ Preimage of a compact set may NOT be cmpt. (often closed & bounded but not cmpt)

Uniform continuity: $f: X \to Y$ is unif cont. if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall p, q \in X$, $d_X(p, q) < \delta \Rightarrow d_Y(f(p), f(q)) < \varepsilon$

→ here one δ works $\forall p \in X$

ex : $\sin(x) : \mathbb{R} \to \mathbb{R}$ uniformly cont

→ f is cont & compact → f is unif cont

$\because \exists N_1 > 0$ s.t. $\forall n \in A$, $n > N_1$, $d(f(p_n), f(p)) < \frac{\varepsilon}{3}$

$\because \exists N_2 > 0$ s.t. $\forall n \in A$, $n > N_2$ $d(f(q_n), f(p)) < \frac{\varepsilon}{3}$

$\therefore \forall n > \max(N_1, N_2)$ $d(f(p_n), f(q_n)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3} < \varepsilon$
↳ contradicts $d(f(p_n), f(q_n)) > \varepsilon$

→ $S \subset X$: $S \to Y$ also unif cont. (same w/ cont)

Thrm : $f: X \to Y$ cont, X is compact, and f is a bijection then $f^{-1}: Y \to X$ is continuous

→ $h^{-1}(E)$ is closed $= f(E)$, but if X not compact then conc fails

Connectedness: X is connected iff the only subset of X that is both open & closed are X and $\emptyset$

Thrm: If $f: X \to Y$ is cont, X is connected, then $f(X)$ connected

$[0, 1] \subset \mathbb{R}$ connected $E \subset X \to f(E)$ conn ↳ as subset of Y w/ induced topology
↳ conn

→ S cannot be written as $A \cup B$ where $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$ (closure in X)

→ $S \subset X$ subspace

if S open in X then $U \subset S$ is open in S iff U is open in X

if S is closed in X then $U \subset S$ is closed in S iff U is closed in X

→ if E is connected $\iff \forall x, y \in E$ and $x < y$ → $[x, y] \subset E$

IVT: If $f : [a, b] \to \mathbb{R}$ cont and $f(a) < f(b)$ then $\forall y \in (f(a), f(b))$ $\exists$ $x \in (a, b)$ s.t. $f(x) = y$

- $[a, b]$ connected $\Rightarrow$ $f([a, b])$ is connected

Discontinuities

(L/R limits): $f(a, b) \to \mathbb{R}$ $x_0 \in (a, b)$

$f(x_0^+) = y$ if $\lim\limits_{t \to x_0} f\big|_{(x_0, b)}(t) = y$

$f(x_0^-) = y$ if $\lim\limits_{t \to x_0} f\big|_{(a, x_n)}(t) = y$

continuous at x_0 iff $f(x_0) = f(x_0^+) = f(x_0^-)$

$$f(x) = \begin{cases} \frac{1}{n} & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

$\forall x \in \mathbb{R} \setminus \mathbb{Q}$, f is continuous

$\forall x \in \mathbb{Q}$ f has simple discont

Monotone functions: $f(a, b) \to \mathbb{R}$, f is weakly monotone inc if $\forall a < x < y < b$ $f(x) \leq f(y)$

Thrm: if $f : (a, b) \to \mathbb{R}$ is monotone increasing then $f(x^+)$ and $f(x^-)$ exist at every $x \in (a, b)$

$f(x^-) = \sup\{f(t) \mid t \in (a, x)\}$

$f(x^+) = \inf\{f(t) \mid t \in (x, b)\}$

if $x < y$ then

$f(x^+) = \inf\{f(t) \mid x < t < y\}$

$\leq \sup\{f(t) \mid x < t < y\} = f(y)$

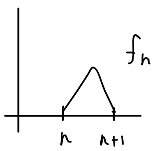
→ has countably many discontinuities → countable disjoint union

Convergence: $f_n : X \to Y$ for $n \in \mathbb{N}$ be a sequence of maps $(f : X \to Y)$

f_n converges to f pointwise $f_n \to f$ if $\forall x \in X$ $\lim\limits_{n \to \infty} f_n(x) = f(x)$

$f_n(x) = 1 + \frac{1}{n}\sin(x)$ $\lim f_n(x) = 1$ (const function)

→ bump function f_n pointwise convergence to 0 → since bump moves



$f_n(x)$ } 0 outside of bump   Supporeted on $[n, n+1]$ interval

→ pointwise limit of a function does NOT preserve integral

→ if $f_n \to f$ pointwise f_n' may not converge to f'

Convergence of objects

sequence of $\vec{x}_n$ (a) $d_2(x,y) = \sqrt{\sum (x_i - y_i)^2}$

$\vec{x}_n \to \vec{x}$ if $\lim_{n\to\infty} d(x_n, x) = 0$

(b) point/ component wise

$$\left.\begin{aligned} x_1 &= (x_{11}, x_{12}, \dots x_{1n}) \\ x_2 &= (x_{21}, x_{22}, \dots\dots) \\ &\vdots \\ x_n &= (x_{n1}, x_{n2}, \dots) \end{aligned}\right\} \text{sequence of } x_n \text{ vectors in } \mathbb{R}^d$$

$x = (x_1, x_2, \dots)$ componentwise

→ $(\vec{x}_n)$ converges to x in a d_2 metric sense $\Longleftrightarrow$ $(\vec{x}_n)$ converges to $\vec{x}$ componentwise

$d_2(x,y) \to d_\infty(x,y) := \max\{|x_i - y_i| : i \in [d]\}$

↳ still convergence $\Longleftrightarrow$ componentwise convergence

(c) $\text{Map}(X, \mathbb{R})$ $\forall i \in \mathbb{N}$ $x_{ni} \to x_i$ (pointwise)

d_2 $\vec{x}_n \to \vec{x}$ $\lim_{n\to\infty} d_2(\vec{x}_n, \vec{x}) = 0 = \sqrt{\sum |x_{ni} - x_i|^2}$

d_∞ $\vec{x}_n \to \vec{x}$ if $\lim_{n\to\infty} d_\infty(\vec{x}_n, \vec{x}) = 0 = \sup\{|x_{ni} - x_i| : i \in \mathbb{N}\}$

Ex: $x_{ni} = \frac{i}{n+i}$ for fixed n $\lim_{i\to\infty} x_{ni} = \lim_{i\to\infty} \frac{i}{n+i} = 1$

for fixed i $\lim_{n\to\infty} \frac{i}{n+i} = 0$

pointwise convergence $= 0$ → $\lim_{n\to\infty} \forall i = 0$

d_2-convergence $\quad d_2(\vec{x_n}, 0) = \sqrt{\Sigma\, x_{ni}^2}$, $\quad \lim_{i\to\infty} x_{ni} = 1 \quad \Sigma\, x_{ni}^2 = \infty$

does not converge to 0

d_∞-sense : $\quad d_\infty(\vec{x_n}, 0) = \sup\{|x_{ni} - 0| : i \in \mathbb{N}\} = 1 \quad \to$ stays constant

Uniform Convergence

$f_n : X \to \mathbb{R}$ a sequence of functions

$f_n \to f$ uniformly if $\forall > 0$, $\exists N > 0$ s.t. $\forall n > N$, $\forall x \in X$

$|f_n(x) - f(x)| < \varepsilon \iff \lim_{n\to\infty} d_\infty(f_n, f) = 0$

only depends on ε not x

$\to x_n$ convergent $\iff x_n$ satisfies Cauchy Condition

Thrm Uniform Cauchy $\iff$ Uniform Convergence

$|f_n(x) - f_m(x)| < \varepsilon \quad \Rightarrow$ unif convergent

- since cauchy $\to \lim_{n\to\infty} f_n(x)$ exists (cauchy $\iff$ conv. for sequence of #)
- $\lim_{m\to\infty} f_m(x) = f(x) \to \lim_{m\to\infty} |f_n(x) - f_m(x)| = |f_n(x) - f(x)| \le \varepsilon$

Thrm : $f_n : X \to \mathbb{R}$ is a sequence of functions and $0 \le M_n \in \mathbb{R}$ s.t.

$M_n \ge \sup_{x\in X} |f_n(x)|$

If $\sum M_n < \infty$, then $\sum f_n$ converges uniformly

$|\sum_{n=N_1}^{N_2} f_n(x)| \le \sum_{n=N_1}^{N_2} |f_n(x)| \le \sum_{N_1}^{N_2} M_n \to$ cauchy test for $\sum M_n$

$\to$ unif cauchy test for $\sum f_n$

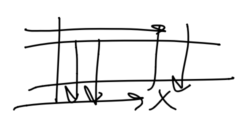
- uniform convergence preserves continuity

Thrm : Let $f_n : E \to \mathbb{R}$ be a sequence of continuous functions, $f_n \to f$ unif on E

Let $x \in E'$ be a limit point, $A_n = \lim_{t\to x} f_n(t)$

① $\lim_{n\to\infty} A_n$ exists $= A$ ← sequence of A_i's which respective f_i's converge too

② $\lim_{t\to x} f(t) = A$



Thrm $f_n : X \to \mathbb{R}$ cont funcs, if $f_n \to f$ uniformly, f is continuous

→ if pointwise convergence → not enough

Thrm : Let K be compact ms $f_n : K \to \mathbb{R}$

assume f_n is continuous $\forall n$

$f_n(x) \to f(x)$ $\forall x \in K$ and f is cont.

$f_n(x) \geq f_{n+1}(x)$ $\longrightarrow$ Then $f_n \to f$ uniformly

Differentiation : $f : [a,b] \to \mathbb{R}$, f is differentiable at point $p \in [a,b]$ if $\lim_{x \to p} \frac{f(x) - f(p)}{x - p}$ exists $(f'(p))$

→ if diffable @ p, then continuous @ p

→ $\exists\, u(x)$ s.t. $f(x) = f(p) + (x-p) \cdot f'(p) + \underbrace{(x-p)\,u(x)}_{\text{remainder}}$ $\quad \lim_{x \to p} u(x) = 0$

→ if R/L limit does not exist then $f'(p)$ does not exist

→ f cont, $f'(x)$ exists $\neq f'$ continuous

chain rule: $(f \cdot g)'(p) = f'(p)\,g(p) + f(p)\,g'(p)$
$(f/g)'(p) = \frac{f'g - f \cdot g'}{g^2}$

$h(x) = g(f(x))$
$h'(x) = g'(f(x)) \cdot f'(x)$

let $f : [a,b] \to \mathbb{R}$, $p \in [a,b]$ is a local maximum of f if there is a $\delta > 0$ s.t. $\forall x \in [a,b] \cap B_\delta(p)$, $f(x) \leq f(p)$

→ locally constant = local max & min

Thrm : $f : [a,b] \to \mathbb{R}$. If p is a local maximum $p \in (a,b)$ and $f'(p)$ exists then $f'(p) = 0$

→ @ endpoints, absolute value turns → $f'(p)$ is non-existant (despite local max)

(Rolle): $f : [a,b] \to \mathbb{R}$ continuous function assume $f'(x)$ exists $\forall x \in (a,b)$

if $f(a) = f(b)$ then $\exists\, c \in (a,b)$ s.t. $f'(c) = 0$

Generalized MVT If $f, g : [a,b] \to \mathbb{R}$ are continuous and diffable on (a,b)

then $\exists\, c \in (a,b)$ s.t.

$$[f(b)-f(a)]\,g'(c) = [g(b)-g(a)]\cdot f'(c)$$

MVT : $f:[a,b]\to\mathbb{R}$ diffable on (a,b) then $\exists\ c\in(a,b)$ s.t.

$$f(b)-f(a) = (b-a)\cdot f'(c)$$

→ proved using lemma that if c is local max/min → derive exists at c and if c is interior point ⇒ derivative vanishes

→ if $f'(x)>0\ \ \forall x\in(a,b)$ then f is strictly increasing function on $[a,b]$

Thrm : $f:\mathbb{R}\to\mathbb{R}$ is continuous and $f'(x)$ exists $\forall x\in\mathbb{R}$

Assume $\exists M>0$ s.t. $|f'(x)|\le M$. $\forall x$ then f is uniform cont

IVT (for $f'(x)$) $f:[a,b]\to\mathbb{R}$ be diffable function $f'(a)<f'(b)$

then for any $\mu\in\mathbb{R}$, with $f'(a)<\mu<f'(b)$, $\exists\ c\in(a,b)$ s.t $f'(c)=\mu$

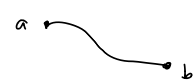
→ prove using $g(x)=f(x)-\mu\cdot x$

L'hopital Rule : if
$$\left.\begin{aligned} f(x) &= (x-a)^h\cdot h(x)\\ g(x) &= (x-a)^h\cdot K(x)\end{aligned}\right\}\ \lim_{x\to a}\frac{f(x)}{g(x)} = \frac{h(a)\neq 0}{K(a)\neq 0}$$

$$\lim_{x\to 0}\frac{\sin(x)}{x} = \lim_{x\to 0}\frac{\sin'(x)}{x'} = \frac{\cos(0)}{1} = 1$$

$f'(a)=f'(b)=0$ can $\exists$ $c\in(a,b)$ s.t $f'(c)=0$

non example : $\left.\lim_{x\to 0}\frac{\log(x)}{x} \longrightarrow \lim_{x\to 0}\log x\neq 0\right\} = -\infty$ not ∞



$f'(c)<0$

No

Thrm : $f,g:(a,b)\to\mathbb{R}$ diffable, $g(x)\neq 0$

(1) $\displaystyle\lim_{x\to a}\frac{f'(x)}{g'(x)} = C \in \mathbb{R}\vee\{+\infty,-\infty\}$

then one of the following holds

$\displaystyle\lim_{x\to a} f(x)=0,\ \lim_{x\to a} g(x)=0$ | $\displaystyle\left.\lim_{x\to a} g(x)=+\infty\right\}\ \lim_{x\to a}\frac{f(x)}{g(x)} = C$

Ex $\displaystyle\lim_{x\to\infty}\left(\frac{x^2}{e^{3x}}\right) = \lim\frac{2x}{3e^{3x}} = \lim\frac{2}{9e^{3x}} = 0$ ✓

Taylor series

(Higher derivatives): $f[a,b] \to \mathbb{R}$ has derivatives $\forall\ x \in [a,b]$ and if $f(x)$ is diffable at all pts $\to f''(x) = (f')'(x)$

n-th derivative $f^{(n)}(x) = \left(f^{(n-1)}\right)'(x)$

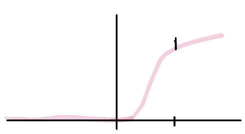
→ smooth functions $f^{(n)}(x)$ exist $\forall\ n \in \mathbb{N}$

→ $\lim_{u\to\infty} u^k \cdot e^{-u} = 0$, $\lim_{x\to 0^+} \frac{1}{x^2} e^{-\frac{1}{x}} = 0$

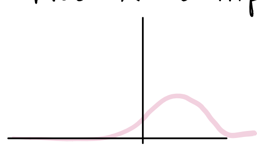
→ prove $f^{n}(x) = \begin{cases} 0 & x \le 0 \\ P_n\left(\frac{1}{x}\right)\cdot e^{-\frac{1}{x}} & x > 0 \end{cases}$ prove by induction

→ if $\alpha(x), \beta(x)$ are smooth on $\mathbb{R}$ and $B(x) \neq 0$ then $\alpha(x)/B(x)$ smooth

smooth step



smooth bump



construct by merging 2 smooth functions

& rescale to fit over $[0,1]$

$g(x) = \dfrac{f(x)}{f(x) + f(1-x)}$

$g(x) = \dfrac{f(x)\,f(1-x)}{f(x) + f(1-x)}$

<u>Thrm</u> $f:[a,b] \to \mathbb{R}$ s.t. $f^{(n-1)}$ exists and is continuous on $[a,b]$ and $f^{(n)}x$ exists on (a,b). Then for any $\alpha, B \in [a,b]$

$$f(\beta) = f(\alpha) + f'(\alpha)\cdot(\beta-\alpha) + \frac{f''(\alpha)}{2!}(\beta-\alpha)^2 + \dots + \frac{f^{(n-1)}(\alpha)}{(n-1)!}(B-\alpha)^{n-1} + R_n(\alpha,\beta)$$

$$R_n(\alpha, B) = \begin{cases} 0 & \text{if } \alpha = B \\ \frac{f^{n}(\gamma)}{n!}(B-\alpha)^n & \text{if } \alpha \neq B \end{cases}$$

→ mth order taylor expansion at α : $f(x) - P_{\alpha, n-1}(x) = \dfrac{f^{n}(\gamma)}{n!}(x-\alpha)^n$

Describe $f(x)$ near $x=\alpha$ of a smooth function

0^{th} : $P_{\alpha 0}(x) = f(\alpha)$ const

1^{st}: $P_{\alpha,1}(x) = f(\alpha) + f'(\alpha)(x-\alpha)$

m^{th}: use $P_{\alpha,m}(x)$ → m deg poly s.t. $P^{k}_{\alpha,m}(\alpha) = f^{(k)}\alpha$

Error linear approx: $(x-\alpha)^2 \cdot \frac{f''(\gamma)}{2!}$ $\gamma \in (x,\alpha)$

$$\rightarrow f(x) - P_{\alpha,n-1}(x) = \frac{f^{(n)}(\gamma)}{n!}(x-\alpha)^n$$

$$P_{x_0}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

↑ 1. given input $x \in \mathbb{R}$ series may not be convergent

2. even if convergent, may not equal $f(x)$

Power series: Series of form $\sum_{n=0}^{\infty} C_n \cdot (x-x_0)^n$

Radius of convergence $(R) = \sup\{r \geq 0,$ s.t. if $|x-x_0| \leq r$ the series converges$\}$

→ if only at $x = x_0$, $R = 0$

→ converges $\forall x$, $R = \infty$

Thrm: $\alpha = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$ $R = \frac{1}{\alpha}$

if $|x-x_0| < R$ series converges

if $|x-x_0| > R$ series diverges

→ Prove using root test for convergence

Ex $\varphi(x) = \begin{cases} 0 & x \leq 0 \\ e^{-\frac{1}{x}} & x > 0 \end{cases}$ $\varphi^n(0) = 0$ $\forall n$

Taylor series of φ at $x=0$ $P_{x_0}(x) \sum^{\infty} \frac{\varphi^n(0)}{n!}(x)^n = 0 \neq \varphi(x)$

→ if $f(x) \sum C_n(x-x_0)^n$ for $|x-x_0| < R$ then $f(x) + \varphi(x-x_0)$ would have same taylor series at x_0

(Def) Smooth func $f: (a,b) \to \mathbb{R}$ is real analytic if $\forall x_0 \in (a,b)$

$f(x) = \sum_n C_n \cdot (x-x_0)^n$ for some neighborhood of x_0

Riemann integrals

→ compute area of irregular shape, curve

→ cut area under curve into thin strips → approx each by something regular

→ take limit using more & more strips

(Def): Partition P of interval $[a,b]$ is $a = x_0 \le x_1 \le \cdots x_n = b$

$\Delta x_i = x_i - x_{i-1}$

Riemann integral: consider any $f:[a,b] \to \mathbb{R}$ that is bounded (not nec cont)

w/ $P = (a = x_0 \le x_1 \le \ldots \le x_n = b)$

- $U(P,f) = \sum_{i=1}^{n} M_i \cdot \Delta x_i$

  $M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$

  $\Delta x_i = x_i - x_{i-1}$

- $L(P,f) = \sum_{i=1}^{n} m_i \cdot \Delta x_i$

  $m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$

$U(f) = \inf U(P,f)$, $L(f) = \sup L(P,f)$ $\qquad \int_a^b f\,dx$

integratble if $U(f) = L(f)$

→ generalization:

$\alpha = [a,b] \to \mathbb{R}$ monoto inc func

$P = \{a = x_0 \le x_1 \le \cdots \le x_n = b\}$, $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) \ge 0$

$\Rightarrow U(P,f,\alpha)$, $U(f,\alpha)$, ...

if $U(P,\alpha) = L(P,\alpha)$ f is Stieltjes integrable wrt α

$f \in R(\alpha) \to \int f(x)\,d\alpha(x)$

set of R-S integ. func ↗

$\int f(x)\,d\alpha(x) = \sum_{n \in \mathbb{Z}} f(n)$

Thrm: If a partition Q is a refinement of a partition P on $[a,b]$

Then "the approximate integrable bounds get better"

$$L_P \le L_Q \le U_Q \le U_P$$

$$L_P = L(P,f,\alpha)$$

Thrm: $L(f,\alpha) \le U(f,\alpha)$ Prove by common refinement (cauchy condition) : f is integrable w.r.t. α $\iff$ $\forall \varepsilon > 0$, $\exists P$ partition s.t. $U_P - L_P < \varepsilon$

<u>Thrm</u>: f, α fixed

(1) if partition P satisfies $U_P - L_P < \varepsilon$ then any refinement Q satisfies $U_Q - L_Q < \varepsilon$

(2) If $U_P - L_P < \varepsilon$ and $\{s_i \in I_i\}$ and $\{t_i \in I_i\}$ are 2 sets of sample points, then $\sum_{i=1}^{n} |f(t_i) - f(s_i)| \cdot \Delta\alpha_i < \varepsilon$

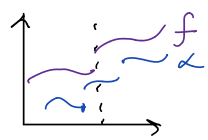
(3) If $U_P - L_P < \varepsilon$ and f integrable and $\{s_i \in I_i\}$
$$\left| \int_a^b f\,d\alpha - \sum f(s_i)\,\Delta\alpha_i \right| < \varepsilon$$

→ If f continuous on $[a,b]$ then f is integrable wrt α on $[a,b]$

$U_P - L_P = \sum_{i=1}^{n} (M_i - m_i) \cdot \Delta\alpha_i$

→ f monotonic and α continuous then $f \in \mathcal{R}(\alpha)$

pf: ① construct evenly dist P_n



② $U_{P_n} - L_{P_n} = \sum_i (M_i - m_i)\,\Delta\alpha_i$

→ if f is discont only at finitely many points and α is continuous where f is discont then $f \in R(\alpha)$

→ $f : [a,b] \to [m, M]$, $\phi : [m, M] \to \mathbb{R}$ is continuous, if f is integrable wrt α then $h = \phi \circ f$ is integrable wrt α

→ integration operation $\int f\,d\alpha$ is linear in both f and α

<u>Thrm</u>: If $f_1, f_2 \in \mathcal{R}(\alpha)$ and $c \in \mathbb{R}$ then

- $f_1 + f_2 \in \mathcal{R}(\alpha)$, $\int f_1 + f_2\,d\alpha = \int f_1\,d\alpha + \int f_2\,d\alpha$
- $c \cdot f_1 \in \mathcal{R}(\alpha)$, $\int c \cdot f_1\,d\alpha = c \cdot \int f_1\,d\alpha$

Linearity in α is similar

If $f, g \in \mathcal{R}(\alpha)$ and $f(x) \leq g(x)$ $\forall x \in [a,b]$ then $\int f\,d\alpha \leq \int g\,d\alpha$

→ If $f, g \in \mathcal{R}(\alpha)$, then $f \cdot g \in \mathcal{R}(\alpha)$

→ if $f \in R(\alpha)$ then $|f| \in R(\alpha)$ and $\left| \int f \cdot d\alpha \right| \leq \int |f| \cdot d\alpha$

→ $f: [a,b] \to \mathbb{R}$ and is continuous at $s \in (a,b)$ and $\alpha(x) = I(x-s)$ then

$$\int f \cdot d\alpha = f(s)$$

→ $c_n \geq 0$ for $n = 1, 2, \ldots$ $\sum c_n < \infty$, $\{s_n\}$ a sequence of distinct points in (a,b)

$$\alpha(x) = \sum_{n=1}^{\infty} c_n \cdot I(x - s_n) = \sum_{n: x > s_n} c_n$$

Let $f: [a,b] \to \mathbb{R}$ be continuous Then $\int f \, d\alpha = \sum_{n=1}^{\infty} c_n \cdot f(s_n)$

Ex $\alpha(x) = I(x)$, $f(x) = I(x) \Rightarrow f$ NOT integrable wrt α

(Streltjes integral): assigns weight to an interval $I: [c, d]$

$\alpha(I) = \alpha(d) - \alpha(c)$. Integral $\int_a^b f \, d\alpha$ is approximated by

"weighted" sum $\sum f(I_i) \cdot \alpha(I_i)$

→ density funtion: $\alpha(I) = \int_c^d \alpha'(x) \, dx \Rightarrow \underset{S-I}{\int f \, d\alpha} = \underset{R-I}{\int f \cdot \alpha' dx}$

→ α monotone inc func on $[a,b]$ s.t. α' exists and is RI

Then for and bounded real function f on $[a,b]$

$$f \in R(\alpha) \iff f \cdot \alpha' \in R$$

$$\int_a^b f \, d\alpha = \int_a^b f \cdot \alpha' \cdot dx$$

Ex $[0,1]$ Let $\alpha = x^2$ then $\alpha' = 2x$ → $f \in R(\alpha)$ iff $f(x) \cdot (2x)$ is integrable

(Change of variables): α increasing on $[a,b]$ $f \in R(\alpha)$

assume strictly increasing, surjective function $\varphi: [A,B] \to [a,b]$

$\beta: [A,B] \to \mathbb{R}$ $\beta = \alpha \circ \varphi$

$g: [A,B] \to \mathbb{R}$ — $g = f \circ \varphi$ (same ruler different marking)

then $g \in R(\beta)$ and $\int_A^B g \, d\beta = \int_a^b f \, d\alpha$

→ f is RI function on $[a,b]$, $F(x) = \int_a^x f(u) \, du$

$F(x)$ is continuous

if $f(x)$ is continuous at x_0, then $F(x)$ is differentiable at x_0, $F'(x_0) = f(x_0)$

(Fundamental Thrm of Calculus): F is differentiable function on $[a, b]$ and $f = F'(x)$ is integrable then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

→ Can have $F(x)$ s.t. $F'(x)$ exists and bounded but $F'(x)$ not integrable

(integration by part): $[a, b]$ assume F, G are differentiable func w/ $F' = f$, $G' = g$ integrable, then

$$\int_a^b F\,dG = F \cdot G \Big|_a^b - \int_a^b G \cdot dF$$

→ $$\int_a^b F(x) g(x) \cdot dx = F(b)G(b) - F(a)G(a) - \int_a^b G(x) f(x)\,dx$$

→ Let α be monotone inc fun on $[a, b]$

f_n be seq of func on $[a, b]$, $f \in R(\alpha)$. Assume $f_n \to f$ unif.

Then $f \in R(\alpha)$ and

$$\int_a^b f\,d\alpha = \lim_{n \to \infty} \int_a^b f_n\,d\alpha$$

→ if $\{f_n\}$ seq of integrable fun wrt α and $\sum_{n=1}^{\infty} f_n(x)$ converges unif

$$\int_a^b \sum_{n=1}^{\infty} f_n(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx$$

→ f_n sequence of differentiable func on $[a, b]$

w/ ① $\exists\, x_0 \in [a, b]$: $f_n(x_0) \to C$

② $\{f_n'(x)\}$ converges uniformly on $[a, b]$

Then $f_n \to f$ uniformly and $f'(x) = \lim_{n \to \infty} f_n'(x)$ $\forall x \in [a, b]$