last lec covered ↓ 2/11/21

Midterm #1 Math 104 Notes (start → midterm)

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1 §1 The Set $\mathbb{N}$ of Natural Numbers

$\{1,2,3,4,\ldots\}$ of all positive integers = $\mathbb{N}$

Properties of $\mathbb{N}$

1) 1 belongs to $\mathbb{N}$
2) if n belongs to $\mathbb{N}$, then its successor $n+1$ belongs to $\mathbb{N}$
3) 1 is not the successor of any element in $\mathbb{N}$
4) If n and m in $\mathbb{N}$ have the successor, then $n=m$
5) A subset of $\mathbb{N}$ which contains 1, and which contains $n+1$ whenever it contains n, must equal $\mathbb{N}$.

Mathmatical induction

Basis for induction: P_1 is true

Induction Step: P_{n+1} is true whenever P_n is true.

examples: "Prove for" - positive integers n
- all numbers are divisible by
- all pos integers n and reel #'s r.

"Integers"

Pg 6 - 13

§2 The Set $\mathbb{Q}$ of Rational Numbers

$\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3 \ldots\}$ = all integers

$\mathbb{Q}$ = all rational #'s $= \frac{m}{n}$ where $m, n \in \mathbb{Z}$ $n \neq 0$

* $\mathbb{Q}$ contains all terminating decimals $1.492 = \frac{1492}{1000}$

$\Rightarrow$ also $(\equiv)$ $nx - m = 0$

1.2.1 Definition Algebraic Number

if it satisfies a polynomial equation

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0$$

where coefficients $c_0, c_1, \ldots c_n$ are integers, $c_n \neq 0$ $n \geq 1$

1.2.2 Rational Zeros Theorem.

Suppose $c_0, c_1, \ldots, c_n$ are integers and r is a rational number satisfying the polynomial equation. $n \geq 1$, $c_n \neq 0$, $c_0 \neq 0$ $r = c/d$ where c and d are integers w/ no common factors $d \neq 0$ Then c divides c_0 and d divides c_n.

Solutions in form $\frac{c}{d}$ where c divide c_0 & d divides c_n.

1.2.3 Corollary:

$$x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0$$

Any rational solution must be an integer that divides c_0.

Pg 13-19

1 §3 The Set $\mathbb{R}$ of Real Numbers.

Properties of $\mathbb{Q}$) Give a pair of rational #'s a,b,
$a+b$ = a rational # $a \cdot b$ = a rational #

A1. $a+(b+c)=(a+b)+c$ for all a,b,c
A2. $a+b=b+a$ for all a,b.
A3. $a+0=a$ for all a.
A4 For each a, there is an element $-a$ such that $a+(-a)=0$
M1 $a(bc)=(ab)c$ for all a,b,c
M2 $ab=ba$ for all a,b.
M3 $a \cdot 1=a$ for all a.
M4 For each $a \neq 0$ there is an element a^{-1} such that $aa^{-1}=1$
DL $a(b+c)=ab+ac$ for all a,b,c.

$\mathbb{Q}$ has the oreder structure $\leq$ satisfying
O1. Give a and b, either $a \leq b$ or $b \leq a$
O2. If $a \leq b$ and $b \leq a$ then $a=b$
O3. if $a \leq b$ and $b \leq c$ then $a \leq c$.
O4. If $a \leq b$ then $a+c \leq b+c$
O5 if $a \leq b$ and $0 \leq c$ then $ac \leq bc$.
=> if a field satisfys O1. → O5. it is an ordered Feild

$\mathbb{R}$ = real numbers; all rational, algebraic numbers, π, e and more
=> $\mathbb{R}$ does not have "gaps" like $\mathbb{N}$, $\mathbb{Q}$
=> like $\mathbb{Q}$ $\mathbb{R}$ is an oredered feild and follows $A_1 \to A_4$, $M_1 \to M_4$, DL, $O_1 \to O_5$

Theorem 1.3.1: consequences of the feild properties
(i) $a+c=b+c$ implies $a=b$
(ii) $a \cdot 0=0$ for all a
(iii) $(-a)b=-ab$ for all a,b
(iv) $(-a)(-b)=ab$ for all a,b
(v) $ac=bc$ and $c \neq 0$ imply $a=b$
(vi) $ab=0$ implies either $a=0$ or $b=0$; for $a,b,c \in \mathbb{R}$

Theorem 1.3.2

(i) If $a \le b$ then $-b \le -a$

(ii) If $a \le b$ and $c \le 0$ then $bc \le ac$

(iii) if $0 \le a$ and $0 \le b$ then $0 \le ab$

(iv) $0 \le a^2$ for all a.

(v) $0 < 1$

(vi) if $0 < a$, then $0 < a^{-1}$

(vii) If $0 < a < b$ then $0 < b^{-1} < a^{-1}$

for $a, b, c \in \mathbb{R}$

$a < b \Leftrightarrow a \le b$ and $a \ne b$

Definition 1.3.3) we define,
$|a| = a$ if $a \ge 0$ and $|a| = -a$ if $a \le 0$
$|a|$ is called the absolute value of a.

Definition 1.3.4) For numbers a and b we define
$\text{dist}(a,b) = |a-b|$; $\text{dist}(a,b)$ represents the distance between a and b

Thm 1.3.5 Properties of Absolute Value

(i) $|a| \geq 0$ for all $a \in \mathbb{R}$

(ii) $|ab| = |a| \cdot |b|$ for all $a, b \in \mathbb{R}$

(iii) $|a+b| \le |a| + |b|$ for all $a, b \in \mathbb{R}$

1.3.6 Corollary) $\text{dist}(a,c) \le (a,b) + \text{dist}(b,c)$ for all $a, b, c \in \mathbb{R}$

1.3.7 Triangle Inequality)

$|a+b| \le |a| + |b|$ for all a, b.

$\Leftrightarrow ||a| - |b|| \le |a-b|$ for all $a, b \in \mathbb{R}$

Pg20-28

1§4 The Completeness Axiom

1.4.1 Definition let S be a nonempty subset of $\mathbb{R}$
(a) If S contains a largest element S_0 [that is, S_0 belongs to S and $s \leq s_0$ for all $s \in S$], then we call S_0 the maximum of S, and write $s_0 = \max S$.

(b) If S contains a smallest element, then we call the smallest element the minimum of S and write it $\min S$

$a<b$

$\Rightarrow \max[a,b] = b$, $\min[a,b] = a$, (a,b) no max no min, $[a,b)$ no max, $(a,b]$ no min

$\Rightarrow$ $\mathbb{Z}$ and $\mathbb{Q}$ have no max or min

$\Rightarrow$ $\mathbb{N}$ has no max but $1 = \min \mathbb{N}$

** some sets don't have max or mins

1.4.2 Definition let S be a nonempty subset of $\mathbb{R}$
(a) If a real number M satisfies $s \leq M$ for all $s \in S$, then M is called an upper bound of S and the set S is said to be bounded above.

(b) If a real number m satisfies $m \leq s$ for all $s \in S$, then m is called the lower bound of S and the set S is said to be bounded below.

(c) The set S is said to be bounded if it is bounded above and below. Thus S is bounded if there exists real numbers m and M such that $S \subseteq [m, M]$

Good examples on p. 22-23

1.4.3 Definition let S be a nonempty subset of $\mathbb{R}$

(a) If S is bounded above and S has a least upper bound, then we will call it the supremum of S denoted $\sup S$.

(b) If S is bounded below and S has a greatest lower bound then we will call it the infimum of S and denote it by $\inf S$.

unlike $\max S / \min S$ $\sup S / \inf S$ need not belong to S.

if S is bounded above, then $M = \sup S$ iff
(i) $s \leq M$ for all $s \in S$
(ii) whenever $M_1 < M$, there exists $s_1 \in S$ s.t. $s_1 > M_1$

1.4.4 | Completeness Axiom |
every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound. In otherwords, $\sup S$ exists and is a real number.

$\Rightarrow$ doesn't hold for $\mathbb{Q}$.

1.4.5 Corollary
Every nonempty subset S of $\mathbb{R}$ that is bounded below has a greatest lower bound $\inf S$.

1.4.6 | Archimedean Property
if $a > 0$, then $\frac{1}{n} < a$ for some int. n.
if $b > 0$, $\Rightarrow$ $b < n$ for some int n.

if $a > 0$ and $b > 0$ then for some positive integer n, we have $na > b$.

1.4.7 Denseness of $\mathbb{Q}$
if $a, b \in \mathbb{R}$ and $a < b$, then there is a rational $r \in \mathbb{Q}$ s.t. $a < r < b$.