Chapter 2 | Sequences

Pg 33-38

2 §7 Limits of Sequences

a sequence is a function whose domain is a set of the form $\{n \in \mathbb{Z} : n \geq m\}$

2.7.1 Definition : converge

a sequence is said to converge to the real number s provided that for each $\varepsilon > 0$ there exists a number N such that $n > N$ implies $|s_n - s| < \varepsilon$

Limit: if (s_n) converges to s, $\lim_{n \to \infty} s_n = s$ or $s_n \to s$.

Diverge: A sequence that doesn't converge, diverges.

$\Rightarrow$ If $\lim s_n = s$ and $\lim s_n = t$ then $s = t$.

Ex seq of irrational #'s with rational limit. $= x_n = \frac{1}{x_n}$.

seq of rational #'s with irrational limits

$r_n = \left(1 + \frac{1}{n}\right)^n$

$* \ \frac{\varepsilon}{2} \Rightarrow$ Page 37 How to slove $\frac{\varepsilon}{2}$ problems.

pg 45–56

2.§9 Limit Theorems for Sequences

2.9.1 Theorem: Convergent sequences are bounded.
be bounded if the set $\{S_n : n \in \mathbb{N}\}$ is a bounded set
$\Leftrightarrow$ if there exists a constant M such that
$|S_n| \leq M$ for all n.

2.9.2 Theorem) If the sequence (S_n) converges to s and k is in $\mathbb{R}$ then the sequence $(K S_n)$ converges to ks, That is, $\lim(KS_n) = k \cdot \lim s_n$

2.9.3 Theorem) If (S_n) converges to s and (t_n) converges to t, then $(S_n + t_n)$ converges to $s+t$. That is

$$\lim (S_n + t_n) = \lim S_n + \lim t_n$$

2.9.4 Theorem) If (S_n) converges to s and (t_n) converges to t, then $(S_n t_n)$ converges to st.
That is,

$$\lim (S_n t_n) = (\lim S_n)(\lim t_n)$$

2.9.5 lemma)
If (S_n) converges to s, if $S_n \neq 0$ for all n, and if $s \neq 0$, then $(1/S_n)$ converges to $1/s$.

2.9.6 Theorem) Suppose (S_n) converges to s and (t_n) converges to t. If $s \neq 0$ and $S_n \neq 0$ for all n, then (t_n/S_n) converges to t/s.

$\Rightarrow$ ex a) $\lim\limits_{n\to\infty} \left(\frac{1}{n^p}\right) = 0$ for $p > 0$
b) $\lim\limits_{n\to\infty} (a^n) = 0$ if $|a| < 1$
c) $\lim\limits_{n\to\infty} (n^{1/n}) = 1$
d) $\lim\limits_{n\to\infty} (a^{1/n}) = 1$ for $a > 0$.

2.9.8 Definition

For a sequence (s_n), we write $\lim s_n = +\infty$ provided for each $M > 0$ there is a number N such that $n > N$ implies $s_n > M$.

$\Rightarrow$ sequence diverges to $+\infty$

$\lim s_n = -\infty$ provided for each $M < 0$ there is a number N such that $n > N$ implies $s_n < M$.

2.9.9 Theorem) let (s_n) and (t_n) be sequences such that $\lim s_n = +\infty$ and $\lim t_n > 0$ [$\lim t_n$ can be finite or $+\infty$] Then $\lim s_n t_n = +\infty$

2.9.10 theorem)

For a sequence (s_n) of positive real numbers, we have $\lim s_n = +\infty$ if and only if $\lim \left(\frac{1}{s_n}\right) = 0$

limit Laws

$\lim a_n = a$

$\lim b_n = b$

① $\{a_n + b_n\} \Rightarrow a+b$

$\{a_n - b_n\} \Rightarrow a-b$

② $\{a_n - b_n\} \Rightarrow ab$

③ $\left\{\frac{a_n}{b_n}\right\} \Rightarrow \frac{a}{b}$ if $M \neq 0$

④ c = constant

$\{c a_n\} \Rightarrow c\,a.$

For $+\infty$, $-\infty$ limits

① if $\lim a_n = +\infty \Rightarrow$

$\lim b_n > 0$

$\lim a_n \cdot b_n = +\infty$

② $\lim a_n = +\infty$

only if

$\lim \left(\frac{1}{a_n}\right) = 0$

pg 56 - 66

2 § 10 Monotone Sequences and Cauchy Seq.

to show seq. converge w/out knowing limits.

2.10.1 Theorem) a sequence (s_n) of real numbers is

Increasing Seq: if $s_n \leq s_{n+1}$ for all n => then $s_n < s_m$ when $n < m$

Decreasing Seq: if $s_n \leq s_{n+1}$ for all n

a sequence that is increasing or decreasing is called monotone.

2.10.2 Theorem) All bounded monotone sequences converge.

Proof of 2.10.2 Relys on Axiom 1.4.4

2.10.3 Decimals) real numbers are decimal expansions

=> different decimal expansions can represent the same #.

Nonnegative decimal expansions => every one is shorthand for the limit of a bounded increasing seq. of real #'s.

$K.d_1 d_2 d_3 d_4 \ldots$ K is a nonneg integer $d_j \in \mathbb{N}$

$\Rightarrow s_n = K + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}$

(s_n) is increasing seq of real #'s and is bounded by $(K+1)$

(=) every real number x has at least 1 decimal expansion.

2.10.4 Theorem)

(i) if (s_n) is an unbounded increasing seq then $\lim s_n = +\infty$

(ii) if (s_n) is an unbounded decreasing seq then $\lim s_n = -\infty$

2.10.5 Corollary) if (s_n) is monotone seq, either converges, or diverges to either $\pm\infty$

2.10.6 Definition) let (s_n) be a seq. in $\mathbb{R}$. We define

$\lim\sup(s_n) = \lim_{N\to\infty} \sup\{s_n : n > N\}$

$\lim\inf(s_n) = \lim_{N\to\infty} \inf\{s_n : n > N\}$

=> not bounded above? = $\sup\{s_n : n > N\} = \lim\sup s_n = \infty$

=> not bounded below? = $\inf(s_n : n > N\} = \lim\inf s_n = -\infty$

$\lim\sup \leq \sup\{s_n : n > N\}$

2.10.7 Theorem) Let (s_n) be a sequence in $\mathbb{R}$

(i) if $\lim s_n$ is defined [real #, $\pm\infty$] then
$\liminf s_n = \lim s_n = \limsup s_n$

(ii) if $\liminf s_n = \limsup s_n$, then
$\lim s_n$ is defined and $\lim s_n = \liminf s_n = \limsup s_n$

$\Rightarrow$ if (s_n) converges, then $\liminf s_n = \limsup s_n$

2.10.8 Definition) A sequence (s_n) of real numbers is called a Cauchy Sequence

for each $\varepsilon > 0$ there exists a number N s.t.
$m, n > N$ implies $|s_n - s_m| < \varepsilon$

2.10.9 Lemma) Convergent seq. are Cauchy Seq

2.10.10 lemma) Cauchy seq. are bounded.

2.10.11 Theorem)

A sequence is a convergent sequence iff it is a Cauchy Sequence.

$\Rightarrow$ use to verify convergence w/out limits.

2.10.11 uses $\Rightarrow$ 2.10.7 and 1.4.4

To Prove increasing s_n prove $s_{n+1} \geq s_n$ for all n.

Pg 66-78

2 § 11 Subsequences

2.11.1 Definition suppose $(S_n)_{n \in \mathbb{N}}$ is a sequence.
A subsequence of this seq. is the seq in form $(t_k)_{k \in \mathbb{N}}$ where for each k there is a positive integer n_k s.t.

$$n_1 < n_2 < \dots < n_k < n_{k+1} < \dots$$

and $$t_k = S_{n_k}$$

subset given by $\sigma(k) = n_k$ for $k \in \mathbb{N}$
subseq. of s corresponding to σ is the composite $t = s \circ \sigma$

$$\Rightarrow t_k = t(k) = s \circ \sigma(k) = s(\sigma(k)) = s(n_k) = S_{n_k} \text{ for } k \in \mathbb{N}$$

Theorem 2.11.2 Let (S_n) be a sequence.
(i) if t is in $\mathbb{R}$, then there is a subseq. of (S_n) converging to t if and only if the set $\{n \in \mathbb{N} : |S_n - t| < \varepsilon\}$ is infinite for all $\varepsilon > 0$.

(ii) if the seq. (S_n) is unbounded above, it has a subsequence with limit $+\infty$

(iii) Similarly, if (S_n) is unbounded below, a subsequence has the limit $-\infty$

2.11.3 Theorem If the sequence (S_n) converges, then every subspace converges to the same limit

2.11.4 Theorem
Every sequence (S_n) has a monotonic sequence.

2.11.5 Theorem Generalized 2.11.4
Every bounded sequence has a convergent subsequence.

2.11.6 Definition | let (S_n) be a sequence in $\mathbb{R}$. A <u>subsequential limit</u> is any real number or symbol $+\infty$ or $-\infty$ that is the limit of some subsequence of (S_n).

2.11.7 Theorem | Let (S_n) be any sequence.

There exists a monotonic subseq. whose limit = $\limsup(S_n)$ and there exists a montotonic subseq whose limit = $\liminf(S_n)$

2.11.8 Theorem | let (S_n) be any sequence in $\mathbb{R}$, and let S denote the set of subsequential limits of (S_n)

(i) S is nonempty

(ii) $\sup S = \limsup S_n$ and $\inf S = \liminf S_n$

(iii) $\lim s_n$ exists iff S has exactly one element, namely $\lim s_n$.

2.11.9 Theorem | Let S denote the set of subsequential limits of a sequence (S_n).

Suppose (t_n) is a sequence in $S \cap \mathbb{R}$ and that $t = \lim t_n$.

$\Rightarrow$ Then t belongs to S.

<u>In other words</u> $\Rightarrow$ the set S of subsequential limits always contain all limits of sequences <u>from</u> S.

"called closed sets"