1. a) $F(x) = \cos(x) - x$ cont. on $\mathbb{R}$, take $[-\pi, \pi]$

$F(0) = 1 - 0 = 1$ $F(\frac{\pi}{2}) < 0 < F(0)$ $\Rightarrow$ By Rudin 4.23, $\exists x \in [0, \frac{\pi}{2}]$ s.t. $F(x) = 0 \Rightarrow \cos x - x = 0 \Rightarrow \cos x = x$

$F(\frac{\pi}{2}) = -\frac{\pi}{2}$

b) (a_n) convergent?

Bounded since $|\cos^n(x)| \leq 1$.

$\sum a_n$ diverge since for all a_n, $a_n > -1$, $\sum -1$ diverge with n odd $\sum = -1$; n even $\sum = 1$.

2. $L(P,f)$ and $U(P,f)$ is step function.

Hint 1: suffice to show for constant

$\lim_{n\to\infty} \int_a^b c \sin(nx)\, dx = \lim_{n\to\infty} \frac{-c}{n} \cos(nx) \Big|_a^b = \lim_{n\to\infty} \frac{-c(\cos(bx) - \cos(ax))}{n} = 0.$

Fix a $\varepsilon > 0$.

Then since $f \in \mathcal{R}$, $\exists P$ s.t. $0 \leq \int_a^b f\,dx - L(P,f) < \varepsilon$. And $L(P,f) = \int_a^b S(x)\,dx$ for some step function. $S(x)$.

$\left|\int_a^b (f(x) - S(x)) \sin nx\, dx\right| \leq \int_a^b (f(x) - S(x)) |\sin nx|\, dx \leq \int_a^b f(x) - S(x)\, dx < \varepsilon.$

$\Rightarrow -\varepsilon + \int_a^b S(x) \sin nx\, dx \leq \int_a^b f(x) \sin nx\, dx \leq \varepsilon + \int_a^b S(x) \sin nx\, dx$

take lim w.r.to $n \to \infty$, then $-\varepsilon \leq \lim_{n\to\infty} \inf \int_a^b f(x) \sin nx\, dx \leq \lim_{n\to\infty} \sup \int_a^b f(x) \sin nx\, dx \leq \varepsilon$. $\forall \varepsilon > 0$

Hence $\lim_{n\to\infty} \sup \int_a^b f(x) \sin nx\, dx = \lim_{n\to\infty} \inf \int_a^b f(x) \sin nx\, dx = \lim_{n\to\infty} \int_a^b f(x) \sin nx\, dx = 0$.

3. a) $F(x) = f(x)(g(a)h(b) - h(a)g(b)) - g(x)(f(a)h(b) - h(a)f(b)) + h(x)(f(a)g(b) - g(a)f(b))$

↳ linear combination of f, g, h, hence F cont. on $[a,b]$, diff. on (a,b)

b) $F(a) = F(b)$, then by Rolle's Thm, $\exists x \in (a,b)$, s.t. $F'(x) = 0$.

c) Consider $h(x) = 1$ and apply part b to it.

4. a) Consider any Cauchy sequence in $(B(x), d_B)$, $\{f_n\}$

By def of Cauchy, $\forall \varepsilon > 0$, $\exists N > 0$, $\forall n, m > N$, $d_B(f_n, f_m) < \varepsilon$. $\Rightarrow \sup_{x \in X} |f_n(x) - f_m(x)| < \varepsilon \Rightarrow |f_n(x) - f_m(x)| < \varepsilon$ for all $x \in X$.

Hence $\{f_n\}$ is uniformly cauchy. implies $\{f_n\}$ uniformly converge to some f.

And since $f_n \to f$ uniformly. then f should also be bounded hence in $B(x)$.

b) By part a, we know all cauchy hence convergent sequences in $B(x)$ converge to some elements in $B(x)$.

Since $C(x) \subset B(x)$, and by uniform convergence. If $\{f_n\} \subset C(x)$ and converge to f, $f \in C(x)$ since f is cont.

By Prop 13.9 (b), $C(x)$ is closed set in $B(x)$.

c) Suppose A is a complete metric space, then any cauchy sequence converges to some $a \in A$. Let B be a closed subset of A.

Then by closeness of B, B contains limits of all convergent sequences in B, which is also cauchy. Hence B contains limit for all cauchy seq. in B.

Thus B is also a complete metric space.

Hence $C(x)$ is a complete metric space.

5. Suppose $[a,b]$ is of measure zero i.e. $\sum \text{vol}(U_k) < \varepsilon$ for all $\varepsilon > 0$. Then it should have a subcover of bad covering.

First note that by Heine-Borel in $\mathbb{R}$, $[a,b]$ is closed and bounded $\Rightarrow [a,b]$ is compact.

Suppose $\exists$ a bad covering for $[a,b]$, $\{B_1, \ldots\}$, then by compactness of $[a,b]$ in $\mathbb{R}$, then

$\{B_1, \ldots, B_n\}$ is a finite bad covering for $[a,b]$. $\sum_{i=1}^n \text{length}(B_i) < (b-a)$.

Then take $B_1 \cup B_2 = V$ to replace B_1 and B_2. And $\text{length}(V) \leq \text{length}(B_1) + \text{length}(B_2)$

By induction, there exists a single interval cover $[a,b]$, but its length is smaller than $b-a$, contradiction.

$\Rightarrow [a,b]$ is not of measure zero.

6. Consider $\varepsilon = 1$. By unif. cont. of f, $|f(x) - f(y)| < \varepsilon = 1$ if $|x-y| < \delta$.

Take $x \geq 1$, $d = \lfloor \frac{x-1}{\delta/2} + 1 \rfloor$ means the # of intervals with length $\delta/2$ in $[1, x]$. Then by that $|f(x) - f(1)| < \lfloor \frac{x-1}{\delta/2} + 1 \rfloor \cdot 1$

$\Rightarrow |f(x) - f(1)| < \frac{x-1}{\delta/2} + 1$

$|f(x)| \leq f(1) + 1 + \frac{x-1}{\delta/2} = ax + b$

Find some Mx such that for all $x \geq 1$, $Mx \geq ax + b$

Then $|f(x)| \leq Mx \Rightarrow \frac{|f(x)|}{x} \leq M$

7. a) Note that $e^{-nx} \cos nx \leq e^{-nx}$

So just consider e^{-nx}.

If $x \in (0, \infty)$, $0 \leq e^{-x} < 1$, hence $\sum_{n=1}^{\infty} \frac{1}{e^{nx}} = \frac{1}{e^x} + \frac{1}{e^x} \cdot \frac{1}{e^x} + \ldots$, by convergence of geometric series, $\sum_{n=1}^{\infty} e^{-nx}$ converges.

If $x = 0$, $e^{-nx} = 1$, $\sum_{n=1}^{\infty} 1 = \infty$, diverges.

If $x \in (-\infty, 0)$, $e^{-nx} = e^{ny}$ for $-x = y$, $y > 0$. Then e^{ny} is not bounded when $n \to \infty$, hence not convergent.

Thus $E = (0, \infty)$

b) No.

8. a) By Rudin 4.29, f is montonic increasing on $(0,1)$, $x \in (0,1)$. $f(x^-) = \sup_{a<t<x} f(t)$, $f(x^+) = \inf_{x<t<b} f(t)$.

Since f is bounded on $(0,1)$ i.e. $t \in (0,1)$, $f(0) \le f(t) \le f(1)$, thus by completeness axiom, sup and inf exists, hence $f(x^-)$, $f(x^+)$ exists

b) By a, $f(x^-)$ and $f(x^+)$ exist at all $x \in (0,1)$.

Then consider $r(x) \in \mathbb{Q}$ s.t. $f(x^-) < r(x) < f(x^+)$.

By 4.29, $f(x^+) \le f(y^-)$ if $x<y$, then $r_1(x) < f(x^+) \le f(y^-) < r_2(x)$. so $r_1(x) \ne r_2(x)$.

Hence a 1-1 correspondence with $\mathbb{Q}$, hence A is at most countable

9. a) If $x \ne 0$, $f(x,y) = \frac{xy}{x^2+y^2}$ for any $y \in \mathbb{R}$.

xy is continuous, x^2+y^2 also continuous and nonzero, then $f(x,y)$ is continuous. (treat x as a constant).

If $x=0$, $f(x,y) = \frac{xy}{x^2+y^2}$ for $y \in \mathbb{R} \setminus \{0\}$.

Similarly $f(x,y)$ is cont.

Check continuity at $y=0$.

$$\lim_{y\to 0^-} f(0,y) = \lim_{x\to 0^+} f(0,y) = \lim_{y\to 0^+} \frac{y}{y^2} \overset{\text{L'hospital}}{=} \lim_{y\to 0^+} \frac{1}{2y} = 0.$$

$\lim_{y\to 0} f(0,y) = f(0,0)$, so $f(0,y)$ is also cont. at $y=0$.

Same for f_y.

b) Consider fixing $x=y$, then $f(x,y) = f(x,x) = \frac{x^2}{2x^2} = \frac{1}{2}$, $\lim_{(x,y)\to(0,0)} f(x,y) = \frac{1}{2} \ne f(0,0)$.

Hence not continuous.

10. $a_{n+1} = \sqrt{a_n^2 + \frac{1}{2^n}}$

$a_{n+1}^2 = a_n^2 + \frac{1}{2^n}$

$a_{n+1}^2 - a_n^2 = \frac{1}{2^n}$

$(a_{n+1}+a_n)(a_{n+1}-a_n) = \frac{1}{2^n}$

Since a_n and a_{n+1} always ≥ 0, and $\frac{1}{2^n} \ge 0$, hence $(a_{n+1}-a_n) \ge 0 \Rightarrow a_{n+1} \ge a_n$

Hence a_n is a monotone increasing sequence.

$a_n = \sqrt{1+\sum_{k=1}^{n-1} \frac{1}{2^k}}$.

$a_1 = 1$ ✓

Ind. Step: $a_n = \sqrt{1+\sum_{k=1}^{n-1} \frac{1}{2^k}}$

$$a_{n+1} = \sqrt{\sqrt{1+\sum_{k=1}^{n-1} \frac{1}{2^k}}^2 + \frac{1}{2^n}}$$

$$= \sqrt{1+\sum_{k=1}^{n} \frac{1}{2^k}}$$

$$\lim_{n\to\infty} a_n = \sqrt{1+\lim_{n\to\infty}\sum_{k=1}^{n} \frac{1}{2^k}} = \sqrt{2}.$$

11. All roots are in $\mathbb{R}$, then we rewrite $P(x) = C(x-r_1)\dots(x-r_n)$ since P only has deg P roots.

Assume $r_1 \le r_2 \le \dots \le r_n$.

Hence $P(r_i) = P(r_{i+1}) = 0$, and P continuous on $\mathbb{R}$, by Rolle's Thm, exist $r_i' \in (r_i, r_{i+1})$ s.t. $P'(r_i') = 0$.

Apply repetitively $n-1$ times, all $r_i' \in \mathbb{R}$, $i = 1, \dots, n-1$.

12. ⌜Fix a $\varepsilon > 0$ and $x_0 \in X$.

Claim: (f_n) converges to 0 pointwise.

$f_n(x_0) \ge 0$ $\forall n \in \mathbb{N}$ and $f_n(x_0) \ge f_{n+1}(x_0)$, (f_n) bounded for a fixed x_0, and monotone decreasing. $\Rightarrow$ (f_n) converges.

By $\lim_{n\to\infty} \sup\{f_n(x) : x \in X\} = 0$, we know for any $\varepsilon > 0$. $\exists N > 0$ s.t. $\sup\{f_n(x) : x \in X\} < \varepsilon$ if $n > N$.

$\Rightarrow f_n(x) < \varepsilon$ for all $x \in X$, $n > N$. Hence $f_n(x_0) < \varepsilon$ for $n > N$. Hence by definition, $\lim_{n\to\infty} f_n(x_0) = 0$. ⌟

Consider this claim, together with $\lim_{n\to\infty} \sup\{f_n(x) : x \in X\} = \lim_{n\to\infty} \sup\{f_n(x) - 0 : x \in X\} = 0$, we conclude $(f_n) \to 0$ uniformly.

Then by alternating series test, $f_n(x) \ge f_{n+1}(x)$ for all $n \in \mathbb{N}$, $x \in X$ and $f_n(x) \ge 0$, $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in X$.

Then $\sum_n (-1)^n f_n(x)$ converges uniformly.

13. f uniform cont. $\Rightarrow |x-y| < \delta \Rightarrow |f(x)-f(y)| < \varepsilon$. Consider $\frac{1}{n} < \delta$, then $|f(\frac{k+1}{n}) - f(\frac{k}{n})| < \varepsilon$ $(\frac{k+1}{n} - \frac{k}{n} = \frac{1}{n} < \delta)$.

Hence for $n = 2m$ for some m:

$$\left|\frac{1}{n}\sum_{k=1}^{n} (-1)^k f(\tfrac{k}{n})\right| < \frac{1}{n} \cdot \frac{n}{2} \cdot \varepsilon < \frac{\varepsilon}{2} < \varepsilon.$$

If $n = 2m+1$:

$$\left|\frac{1}{n}\sum_{k=1}^{n} (-1)^k f(\tfrac{k}{n})\right| < \frac{1}{n} \cdot \frac{n-1}{2} \cdot \varepsilon + \frac{|f(1)|}{n} < \frac{\varepsilon}{2} + \frac{|f(1)|}{n}$$ ← could find some n such that $\frac{|f(1)|}{n} < \frac{\varepsilon}{2}$.

Therefore $|\frac{1}{n}\sum_{k=1}^{n} (-1)^k f(\frac{k}{n})| < \varepsilon$ for $\varepsilon > 0$, hence $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} (-1)^k f(\frac{k}{n}) = 0$.

14. Consider $g(x) = f(x) - f(x+\frac{T}{2})$ is cont.

$g(0) = f(0) - f(\frac{T}{2})$

$g(\frac{T}{2}) = f(\frac{T}{2}) - f(T) = f(\frac{T}{2}) - f(0)$ since $f(0) = f(0+T) = f(T)$

If $g(0) = g(\frac{T}{2}) = 0$, then add T to both, i.e. $g(T)$ and $g(\frac{3T}{2})$, to get 2 non-zero values.

WLOG assume $g(0)$ and $g(\frac{T}{2}) \neq 0$.

$g(0)$ and $g(\frac{T}{2})$ must have different signs, then by Rudin 4.23, $g(0) < 0 < g(\frac{T}{2})$ or $g(\frac{T}{2}) < 0 < g(0)$.

$\exists x_0 \in (0, \frac{T}{2})$ such that $g(x_0) = 0$ i.e. $f(x_0) = f(x_0 + \frac{T}{2})$.

15. Let $f(x) = a_1 \sin x + a_2 \sin 2x + \ldots + a_n \sin nx$

$|f(x)| \leq |\sin x|$

$|f(x) - 0| \leq |\sin x - 0|$ Note: $f(0) = \sin 0 = 0$

$|f(x) - f(0)| \leq |\sin x - \sin 0|$

$$\frac{|f(x) - f(0)|}{|x - 0|} \leq \frac{|\sin x - \sin 0|}{|x - 0|}$$

$$\lim_{x\to 0} \left|\frac{f(x) - f(0)}{x - 0}\right| \leq \lim_{x\to 0} \left|\frac{\sin x - \sin 0}{x - 0}\right|$$

$$\left|\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}\right| \leq \left|\lim_{x\to 0} \frac{\sin x - \sin 0}{x - 0}\right|$$ both diff. at 0.

$|f'(0)| \leq |\cos(0)|$

$|a_1 + 2a_2 + \ldots + na_n| \leq 1$.

16. False. Prof. Fan's eg.

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{\sqrt{p}} \text{ where } p \text{ is prime} \\ 0 & \text{otherwise} \end{cases}$$

Claim: $\{\frac{1}{\sqrt{p}} : p \text{ prime}\} \cap \{\frac{r}{n} : r \in \mathbb{R}, n \in \mathbb{N}\}$ at most once.

If intersect twices, $\frac{1}{\sqrt{p_1}} = \frac{r}{n_1}, \frac{1}{\sqrt{p_2}} = \frac{r}{n_2}$, then $\frac{\sqrt{p_1}}{\sqrt{p_2}} = \frac{n_1}{n_2}$ which is impossible, hence at most one intersection.

Hence $\lim_{n\to\infty} f(\frac{r}{n}) = 0$

But $\lim_{x\to 0} f(x)$ does not exist, if we consider forming a seq. converging to 0 by $\{\frac{1}{\sqrt{p}} : p \text{ prime}\}$, then limit is 1; if form a seq. with element not in $\{\frac{1}{\sqrt{p}}, p \text{ prime}\}$, then limit is 0.

17. Consider $h(x) = f(x) e^{g(x)}$, which is also cont. on $[a, b]$ and diff. on (a, b).

$h(a) = f(a) e^{g(a)} = 0 = h(b) = f(b) e^{g(b)}$

By Rolle's Thm, $\exists x \in (a, b)$ s.t. $h'(x) = 0$ i.e. $f'(x) e^{g(x)} + f(x) e^{g(x)} \cdot g'(x) = 0$

$\Rightarrow e^{g(x)} (f'(x) + f(x) g'(x)) = 0$ since $e^{g(x)} > 0$, hence $f'(x) + g'(x) f(x) = 0$ for some $x \in (a, b)$

18. a) Suppose $f \in \mathcal{R}$, $\exists P = \{0 = t_0 < \ldots < t_\ell = 1\}$ and $U(f, P) - L(f, P) < \varepsilon$.

$R_n = \frac{1}{n} \sum_{k=1}^{n} f(\frac{k}{n})$, each $\frac{k}{n} \in [t_{i-1}, t_i]$, hence $\inf_{[t_{i-1}, t_i]} f(x) \leq f(\frac{k}{n}) \leq \sup_{[t_{i-1}, t_i]} f(x)$

$$\Rightarrow R_n \leq \frac{1}{n} \sum_{i=1}^{\ell} \#\{1 \leq k \leq n \mid \tfrac{k}{n} \in [t_{i-1}, t_i]\} \cdot \sup_{[t_{i-1}, t_i]} f(x) \leq \frac{1}{n} \sum_{i=1}^{\ell} (n(t_i - t_{i-1}) + 1) \cdot \sup f(x)$$

$$= \sum_{i=1}^{\ell} (t_i - t_{i-1}) \sup f(x) + \frac{1}{n} \sum \sup f(x) = U(f, P) + \frac{\ell}{n} \sup f(x)$$

Take $\lim_{n\to\infty}$, then $\limsup R_n \leq U(P, f)$. By similar step, we can get $\liminf R_n \geq L(P, f)$.

Then $\limsup R_n \leq U(P, f) < L(P, f) + \varepsilon \leq \liminf R_n + \varepsilon$. $\forall \varepsilon$, then $\limsup R_n \leq \liminf R_n$, hence $\lim R_n$ exists.

$\lim R_n \leq U(f, P)$ for all P, and $\lim R_n \geq L(f, P)$ for all P, hence $L(f) \leq \lim R_n \leq U(f)$, by $f \in \mathcal{R}$, $L(f) = R(f)$,

thus $\lim R_n = \int_0^1 f\,dx$.

b) Take $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ Not integrable since $U(f) = 1$, $L(f) = 0$

But $\lim R_n = \lim \frac{1}{n} \sum_{k=1}^{n} 1 = \lim 1 = 1$.