§14 - Series

14.2 - Infinite Series

$$\sum_{n=m}^{\infty} a_n = S \Rightarrow \lim S_n = S \Rightarrow \lim_{n\to\infty}\left(\sum_{k=m}^{n} a_k\right) = S$$

$\sum_{n=m}^{\infty} a_n$ converges to S, if it does not converge it diverges to $+-\infty$

Geometric Series

- $\sum_{k=0}^{n} ar^k = a\frac{1-r^{n+1}}{1-r}$ for $r \neq 1$
- $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ if $|r| < 1$
- $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$
  $p \leq 1$ then $\sum \frac{1}{n^p} = +\infty$

14.3 Def $\sum a_n$ satisfies Cauchy Criterion if seq. (S_n) of partial sums is a Cauchy seq.

for each $\varepsilon > 0$ there is a number N s.t. $n \geq m > N$ implies

$$\left|\sum_{k=m}^{n} a_k\right| < \varepsilon.$$

14.4 A series converges iff it satisfies Cauchy Criterion

14.5 Corollary - If a series $\sum a_n$ converges, then $\lim a_n = 0$

$\boxed{14.6}$ **Comparison Test**

Let $\sum a_n$ be a series where $a_n \geq 0$ for $\forall n$.

(i) If $\sum a_n$ converges and $|b_n| \leq a_n$ for $\forall n$, then $\sum b_n$ converges.

(ii) If $\sum a_n = +\infty$ and $b_n \geq a_n$ for $\forall n$, then $\sum b_n = +\infty$

14.7 $\Rightarrow$ Absolutely Convergent series are convergent.

$\boxed{14.8}$ **Ratio Test**

A series $\sum a_n$ of nonzero terms

(i) converges absolutely if $\limsup |a_{n+1}/a_n| < 1$

(ii) diverges if $\liminf |a_{n+1}/a_n| > 1$

(iii) Otherwise $\liminf |a_{n+1}/a_n| \leq 1 \leq \limsup |a_{n+1}/a_n|$ and the test gives <u>no</u> information.

$\Rightarrow$ if $\lim |a_{n+1}/a_n|$ exists $\limsup = \lim = \liminf$.

14.9 **Root Test**

Let $\sum a_n$ be a series and let $\alpha = \limsup |a_n|^{1/n}$

The series $\sum a_n$

(i) converges absolutely if $\alpha < 1$

(ii) diverges if $\alpha > 1$

(iii) otherwise, $\alpha = 1$ and no info given.

Remark on Tests

$\Rightarrow$ if a_n terms are non-zero and $\lim |a_{n+1}/a_n| = 1$ then, $\alpha = \limsup |a_n|^{1/n} = 1$ $\Rightarrow$ No info from Root or Ratio.

if a_{n+1}/a_n is easy to calculate try RATIO TEST

§15 Alternating Series and Integral Tests

15.1 (Theorem) $\sum \frac{1}{n^p}$ converges iff $p > 1$

$\sum \frac{1}{n^p}$ diverges iff $0 < p < 1$

$\Rightarrow$ since $\sum \frac{1}{n}$ diverges $\sum \frac{1}{n^p}$ diverges by comparison

15.2 Integral Tests

Conditions for integral test

(a) Comparison, Root or Ratio test don't apply

(b) The terms of a_n of the series $\sum a_n$ are non-Negative.

(c) There is a nice decreasing function f on $[1, \infty)$ such that $f(n) = a_n$ for all n [f decreasing if $x < y$ implies $f(x) \geq f_y$]

(d) The integral of f is easy to calculate or Estimate

General Integral Test

Let f be a non-negative, continuous function defined on the unbounded interval $[N, \infty)$, N is an arbitrary integer such that f is monotone decreasing on $[N, \infty)$

$\sum_{n=N}^{\infty} f(n)$ converges to a real # iff the improper integral $\int_N^{\infty} f(x)\,dx$ is finite.

15.3 Alternating Series Theorem

If $a_1 \geq a_2 \geq \ldots \geq a_n \cdots \geq 0$ and $\lim a_n = 0$ then, the alternating series $\sum (-1)^{n+1} a_n$ converges.

$\Rightarrow$ Partial sums $s_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$ satisfies $|s - s_n| \leq a_n$ for all n.

$\sum (-1)^n a_n$ is alternating because the signs of terms alternate between + and −.