CH. 7 Rudin

Ross 24, 25, 26

Convergence of Seq. of Functions

Ross ch. 4

§ 24 Uniform Convergence.

24.1 | Def | let (f_n) be a seq. of real valued functions defined on a set $S \subseteq \mathbb{R}$.
Sequence $f(n)$ converges pointwise [at each point] to a function f defined on S if.

$$\lim_{n\to\infty} f_n(x) = f(x), \text{ for } \forall\ x \in S$$

$\Rightarrow$ written as $\lim f_n = f$ pointwise [on S] or
$f_n \to f$ pointwise [on S].

24.2 | Def | let (f_n) be a seq of real valued functions defined on a set $S \subseteq \mathbb{R}$

seq (f_n) converges uniformly on S to a function f defined on S if.
for each $\varepsilon > 0$ there exists a # N such that

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for } \forall x \in S \text{ and } \forall n > N$$

written as $\Rightarrow$ $\lim f_n = f$ uniformly on S
or $\Rightarrow$ $f_n \to f$ uniformly on S

24.3 Theorem) The uniform limit of continuous functions is continuous. (more percise pg. 196)

24.4
Remark: A seq. (f_n) of functions on a set $S \subseteq \mathbb{R}$ converges uniformly to a function f on S iff

$$\lim_{n\to\infty} \sup\{|f(x) - f_n(x)| : x \in S\} = 0$$

$\Rightarrow$ if $(f - f_n)$ is differentiable, use calculus to find suprema.

§25 more uniform Convergence

Discussion (a) if g and h are integrable on $[a,b]$
and if $g(x) \le h(x)$ for $\forall x \in [a,b]$
then, $\int_a^b g(x)\,dx \le \int_a^b h(x)\,dx$

b) If g is integrable on $[a,b]$ then.

$$\left| \int_a^b g(x)\,dx \right| \le \int_a^b |g(x)|\,dx$$

25.2 Theorem | let (f_n) be a sequence of continuous functions on $[a,b]$, and suppose $f_n \to f$ uniformly on $[a,b]$. Then,

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

25.3. Def | A seq. (f_n) of functions defined on a set $S \subseteq \mathbb{R}$ is uniformly Cauchy on S if

for each $\varepsilon > 0$ there exists a ~~#~~ N such that

$$|f_n(x) - f_m(x)| < \varepsilon \text{ for } \forall x \in S \;\&\; \forall m,n > N$$

25.4. Theorem | Let (f_n) be a seq of functions defined and uniformly Cauchy on a set $S \subseteq \mathbb{R}$
Then there exists a function f on S such that $f_n \to f$ uniformly on S.

§25 cont Series of functions

25.5-Theorem | Consider a series $\sum_{k=0}^{\infty} g_k$ of functions on a set $S \subseteq \mathbb{R}$.
Suppose each g_k is continuous on S and the series converges uniformly on S.
Then $\Rightarrow$ series $\sum_{k=0}^{\infty} g_k$ represents a Continuous function on S.

25.6-Thm | If a series $\sum_{k=0}^{\infty} g_k$ of functions satisfies the Cauchy Criterion uniformly on set S, then the series converges uniformly on S.

25.7-Weierstrass M-test | let (M_k) be a seq. of real-non-negative numbers where $\sum M_k < \infty$
If $|g_k(x)| \leq M_k$ for all x in a set S, then

$\sum g_k$ converges uniformly on S.

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7.12
7.13 pg 150

Rudin CH 7 cont

7.11 Theorem | Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E, and suppose that

$$\lim_{t \to x} f_n(t) = A_n \quad (n = 1, 2, 3 \ldots)$$

Then $\{A_n\}$ converges and $\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n$

$$\Longleftrightarrow \quad \lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t)$$

7.12 Thm | If $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \to f$ uniformly on E, then, f is continuous on E.

7.13 Thm | Suppose K is compact and,

(a) $\{f_n\}$ is a seq. of continuous functions on K.

(b) $\{f_n\}$ converges pointwise to a continuous function f on K.

(c) $f_n(x) \geq f_{n+1}(x)$ for $\forall x \in K$ $n = 1, 2, 3 \ldots$

Then $f_n \to f$ uniformly on K.