§13 Topological Concepts in Metric Spaces

13.1 $\boxed{\text{Def}}$ S is a set, d is a function defined for all pairs (x,y) of elements from S satisfying

D1. $d(x,x)=0$ for all $x \in S$ and $d(x,y)>0$ for distinct x,y in S.

D2. $d(x,y)=d(y,x)$ for all $x,y \in S$

= triangle inequality → D3. $d(x,z) \le d(x,y)+d(y,z)$ for all $x,y,z \in S$

This is distance function or a metric on S.

A metric space S is a set S with a metric on it.

⇒ Pair (S,d) because S can have more than 1 metric.

13.2 $\boxed{\text{Def}}$ A sequence (s_n) in a metric space (S,d) converges to s in S if $\lim_{n\to\infty} d(s_n, s) = 0$.

A sequence (s_n) in S is a Cauchy Sequence if for each $\varepsilon>0$ there exists an N s.t. $m,n>N$ implies $d(s_m,s_n)<\varepsilon$

The metric is complete if every cauchy seq. in S converges to an element in S.

NOTATION = To prove $\mathbb{R}^k$ is complete $x_n = x^{(n)}$ to not confuse notation

$$x_n = x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \dots, x_k^{(n)})$$

13.3 $\boxed{\text{Lemma}}$ A sequence $(x^{(n)})$ in $\mathbb{R}^k$ converges iff for each $j=1,2,\dots k$ the seq. $(x_j^{(n)})$ converges in $\mathbb{R}$.

$(x^{(n)})$ is Cauchy in $\mathbb{R}^k$ iff each seq. $(x_j^{(n)})$ is Cauchy seq. in $\mathbb{R}$.

13.4 | Theorem | Euclidean k-space $\mathbb{R}^k$ is complete.

13.5 Bolzano-Weierstrass Theorem.

Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

13.6 | Def | let (S, d) be a metric space.
let E be a subset of S.
An element $s_0 \in E$ is interior to E if for some $r > 0$
$\{ s \in S : d(s, s_0) < r \} \subseteq E$

E° = the set of interior points to E.
if $E^\circ = E$ then the set E is open.

13.7 | Discussion | (i) S is open in S [trivial]
(ii) The empty set $\emptyset$ is open in S [trivial]
(iii) The union of any collection of open sets is open.
(iv) The intersection of finitely many open sets is again an open set.

13.8 | Def | let (S, d) be a metric space.
The subset E of S is closed if its complement $S \setminus E$ is an open set.
⇒ E is closed if $E = S \setminus U$ where U is an open set.

13.7 cont ⇒ the intersection of any collection of closed sets is closed.
⇒ the closure E^- of a set E is the intersection of all closed sets w/ E.
⇒ Boundary of E is set $E^- \setminus E^\circ$
↳ these are boundary points.

13.9 **Proposition** Let E be a subset of metric space (S,d)

(a) The set E is closed iff $E = E^-$

(b) The set E is closed iff it contains the limit of every convergent seq. of points in E.

(c) An element is in E^- iff it is the limit of some seq. of points in E.

(d) A point is in the boundary of E iff it belongs to the closure of both E and its complement.

Ex Open/closed sets

- in $\mathbb{R}$ open interval (a,b) is an open set.
  closed interval $[a,b]$ is a closed set
  The boundary of (a,b) and $[a,b]$ is $\{a,b\}$
- Every open set in $\mathbb{R}$ is the union of a disjoint seq. of open intervals.
  $\Rightarrow$ not applicable for closed sets.
- if $a<b$ then (a,b) or $[a,b]$ cannot be written as the disjoint union of two or more closed intervals, each with at least 1 point.

- open balls are open sets "$<$"
  closed balls are closed sets "$\leq$"
  $\rightarrow$ balls can be neither open or closed.

13.10 **Theorem** (F_n) is a decreasing seq. [ie. $F_1 \supseteq F_2 \supseteq \ldots$] of closed, bounded, nonempty sets in $\mathbb{R}^k$.

Then $F = \bigcap_{n=1}^{\infty} F_n$ is also closed, bounded, nonempty.

13.11 [Def] Let (S,d) be a metric space.

A family $\mathcal{U}$ of open sets is said to be an open cover for a set E if each point of E belongs to at least 1 set in $\mathcal{U}$

i.e. $E \subseteq \bigcup \{U : U \in \mathcal{U}\}$

=> subcover of $\mathcal{U}$ is any subfamily that covers E.

=> if finite if it only contains finitely many sets. the sets themself do not need to be finite.

=> Compact: if every open cover of E has a finite subcover of E.

13.12 [Theorem] Heine-Borel Theorem.

A subset E of $\mathbb{R}^k$ is compact iff it is closed and bounded.

13.13 [Proposition]

Every k-cell F in $\mathbb{R}^k$ is compact.

Rudin
Chapter 2 Topology:

$f(x)$ = an element in B x = an element in A.
Function from A to B (or mapping A into B)
A is the domain
the set $f(x)$ and all it's elements is the range.

If $E \subset A$, $f(E)$ is defined to be the set of all elements $f(x)$ for $x \in E$
$f(E)$ is the image of E under f.

$$\boxed{A} \xrightarrow{f} \boxed{B}$$

if $E \subset A \Rightarrow f(E)$ is defined
$f(E)$ is the image of E under f

$f(A)$ = range of f
$f(A) \subset B$ $\Rightarrow$ if $f(A) = B$ f maps A onto B

if $E \subset B \Rightarrow f^{-1}(E)$ denotes the set of $x \in A$ s.t. $f(x) \in E$ this is the inverse image of E under f.

if $y \in B \Rightarrow f^{-1}(y)$ is set of $x \in A$ s.t. $f(x) = y$
for each $y \in B$, $f^{-1}(y)$ consists of at most 1 element of A.
if so f is 1-1 mapping A into B.
1-1 mapping into
if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ $x_1 \in A$ $x_2 \in A$.
$\boxed{\text{ONTO}}$
$\boxed{\text{Equivalence Relation:}}$ $\Rightarrow$ 1-1 mapping onto
$\Rightarrow$ 1-1 correspondence
1) Reflexive $A \sim A$ $\Rightarrow$ same cardinal #
2) Symmetric if $A \sim B \to B \sim A$ $\Rightarrow$ equivalent. = $A \sim B$
3) transitive
if $A \sim B$, $B \sim C \Rightarrow A \sim C$

Def) for any positive integer n, let J_n be set whose elements are integers $1, 2 \ldots n$ = set J for Any set A.

a) A is finite if $A \sim J_n$ for some n ($\emptyset$ is finite)

b) A is infinite if A is not finite.

c) A is countable if $A \sim J$

d) A is uncountable if A is neither finite or countable.

e) A is at most countable if A is finite or countable.

Countable = enumerable or denumerable.

$\Leftrightarrow$ A is infinite if A is equiv. to one of it's proper subsets

Theorem) Every infinite subset of a countable set A is countable.

A and Ω are sets: for each $\alpha \in A$ there is a subset of Ω denoted E_α

$\{E_\alpha\}$ = the set whose elements are the sets E_α

Union of sets E_α; the set S st. $x \in S$ iff $x \in E_\alpha$ for at least one $\alpha \in A$

$S = \bigcup_{\alpha \in A} E_\alpha \Rightarrow$ if A is integers $S = \bigcup_{m=1}^{n} E_m \Rightarrow$ Positive integers $S = \bigcup_{m=1}^{\infty} E_m$ ∞ means "countable" sets not $\pm\infty$.

Intersection of sets E_α is defined P st. $x \in P$ iff $x \in E_\alpha$ for $\forall \alpha \in A$

$$P = \bigcap_{\alpha \in A} E_\alpha \Rightarrow P = \bigcap_{m=1}^{\infty} E_m$$

if $A \cap B$ is not empty $\Rightarrow$ A and B intersect

otherwise $\Rightarrow$ disjoint.

Ex

a) $E_1 = 1, 2, 3$ $\quad E_1 \cup E_2 = 1, 2, 3, 4$

$E_2 = 2, 3, 4$ $\quad E_1 \cap E_2 = 2, 3$

b) A real #'s x s.t. $0 < x \leq 1$

for $x \in A$, let E_x real #'s y s.t. $0 < y < x$

$A = 0 < x \leq 1$ $\Rightarrow 0 < y < x \leq 1$

$x \in A$ $\quad E_x = 0 < y < x$

$\Rightarrow$ (i) $E_x \subset E_z$ iff $0 < x \leq z \leq 1$

(ii) $\bigcap_{x \in A} E_x = E_1$

(iii) $\bigcap_{x \in A} E_x$ is empty

Union / Intersection Properties

$A \cup B = B \cup A \qquad A \cap B = B \cap A$

$(A \cup B) \cup C = A \cup (B \cup C) \qquad (A \cap B) \cap C = A \cap (B \cap C)$

$\Rightarrow A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \subset A \cup B$ $\qquad \emptyset$ = empty set $\Rightarrow$

$A \cap B \subset A$ $\qquad A \cup \emptyset = A \quad A \cap \emptyset = \emptyset$

if $A \subset B$ then

$A \cup B = B \quad A \cap B = A$

Theorem $\{E_n\}, n = 1, 2, 3 \ldots$, be a seq of countable sets

$S = \bigcup_{n=1}^{\infty} E_n \Rightarrow$ then S is countable

Corollary Suppose A is at most countable, and, for every $\alpha \in A$, B_α is at most countable.

$T = \bigcup_{\alpha \in A} B_\alpha \Rightarrow$ Then T is at most countable.

2.13 Theorem) Let A be a countable set
B_n be the set of all n-tuples
$(a_1, \ldots a_n)$ where $a_k \in A (k=1, \ldots, n)$
elements $(a_1, \ldots, a_n)$ do not need to be distinct

$\Rightarrow$ then B_n is countable.

Corollary The set of all rational #'s is countable.

Thm) Let A be a set of all seq whose elements are didgets 0 and 1. Set A is uncountable.

Metric Spaces

Def metric space if with any two points p and q of X there is a real # d(P,q) called the distance from p to q. s.t.

a) $d(P,q) > 0$ if $p \neq q$; $d(P,P) = 0$
b) $d(P,q) = d(q,P)$
c) $d(P,q) \leq d(p,r) + d(r,q)$ for $\forall\, r \in X$

these are properties for distance function or metric

Metric Spaces R^1 (real line), R^2 (complex plane)
distance in $R^k \Rightarrow d(x,y) = |x-y|$ $(x, y \in R^k)$

segment $= a < x < b$, interval $= a \leq x \leq b$

Def let X be a metric space.
Points/ sets below are elements/subsets of X.

a) a neighborhood of a point p is a set $N_r(P)$ consisting of all the points q s.t. $d(P,q) < r$. r = radius
b) Point p is a limit point of set E if every neighborhood of P contains a point $q \neq P$ s.t. $q \in E$.
c) if $P \in E$ and p is not a limit point, $\Rightarrow$ P is an isolated Point
d) E is closed if every limit point is a point in E.
e) P is an interior point of E if there is a neighborhood N of p s.t. $N \subset E$
f) E is open if every point of E is an interior point of E.
g) Complement (E^c) set of all points $p \in X$ s.t. $p \notin E$
h) E is perfect if E is closed and every point is a limit point of E.

i) E is bounded if a real # M and a point $q \in X$ st $d(P,q) < M$ for $\forall\, p \in E$
(j) E is dense in X if $\forall$ point of X is a limit point, or a point of E (or both)

Thm) Every neighborhood is an open set.

Thm) If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.

Corollary) A finite point set has no limit points.

→ Good examples Pg 33

Thm) Let $\{E_\alpha\}$ be a (finite or infinite) collection of sets E_α. then

$$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha (E_\alpha^c)$$

Thm) A set E is <u>open</u> iff its complement is closed.

<u>Corollary</u>) A set F is <u>closed</u> iff its complement is open.

2.24 Theorem)

a) for any collection $\{G_\alpha\}$ of open sets, $\bigcup_\alpha G_\alpha$ is open.

b) " $\{F_\alpha\}$ of closed sets $\bigcap_\alpha F_\alpha$ is closed

c) for any finite collection $G_1, \ldots G_n$ of open sets, $\bigcap_\alpha F_\alpha$ is closed,

d) " $F_1, \ldots, I_n$ of closed sets, $\bigcup_{i=1}^n F_i$ is closed.

Def) if X is a metric space
if $E \subset X$ and if E' denotes the set of all limit points of E in X, then the closure of E is the set $\bar{E} = E \cup E'$

Thm) if X is a metric space and $E \subset X$ then,
a) $\bar{E}$ is closed
b) $E = \bar{E}$ iff E is closed
c) $\bar{E} \subset F$ for every closed set $F \subset X$ s.t. $E \subset F$

(a) (c) $\Rightarrow$ $\bar{E}$ is the smallest closed subset of X containing E.

Thm) Let E be a nonempty set of real numbers which is bounded above.
Let $y = \sup E$. Then $y \in \bar{E}$. Hence $y \in E$ if E is closed.

Thm) Suppose $Y \subset X$
a subset E of Y is open relative to Y iff $E = Y \cap G$ for some open subset G of X.

Compact Sets. Pg 36 → 43

Def an open cover of a set E is an metric space X we mean a collection $\{G_\alpha\}$ of open subsets of X st $E \subset \bigcup_\alpha G_\alpha$

Def subset K of a metric space X is said to be compact if every open cover of K contains a finite subcover.

if $\{G_\alpha\}$ is an open cover of K, there are finitely many indices $\alpha_1 \ldots \alpha_n$ st
$$K \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$$
$\Rightarrow$ every finite set is compact

Thm) Suppose $K \subset Y \subset X$, Then K is compact relative to X iff K is compact relative to Y.

Thm Compact subsets of metric spaces are closed.

Thm Closed subsets of compact sets are compact.

$\Rightarrow$ Corollary| if F is closed and K is compact, then $F \cap K$ is compact.

Thm If $\{K_\alpha\}$ is a collection of compact subsets of metric spaces X st the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty then $\bigcap K_\alpha$ is non empty.

Corollary | If $\{K_n\}$ is a seq of nonempty compact sets s.t. $K_n \supset K_{n+1}$ $(n = 1, 2, 3 \ldots)$ then, $\bigcap_1^\infty K_n$ is not empty.

Thm | If E is an infinite subset of a compact set K, then E has a limit

Thm | If $\{I_n\}$ is a seq of intervals in R^1, s.t. $I_n \supset I_{n+1}$ $(n = 1, 2, 3 \ldots)$, then $\bigcap_1^\infty I_n$ is not empty.

Thm | let K be a positive integer. If $\{I_n\}$ is a seq of K-cells s.t. $I_n \supset I_{n+1}$ $(n = 1, 2, 3)$, then $\bigcap_1^\infty I_n$ is not empty.

Thm | Every K-cell is compact.

Thm | If a set in R^K has one of the following three properties, that it has the other two.

if a $\Rightarrow$ (b)(c) (a) also (a) E is closed and bounded.
if b $\Rightarrow$ (a)(c) (b) (b) E is compact
c $\Rightarrow$ (a)(b)(c) (c) Every infinite subset of E has a limit point in E.

b ~ c in any metric space
a doesn't in general imply b and c

Thm | (Weierstrass) Every bounded infinite subset of R^K has a limit point in R^K.

Perfect Sets

Thm | Let P be a nonempty perfect set in R^k. Then P is uncountable.

Corrollary | Every interval $[a,b]$ $(a<b)$ is uncountable.
$\Rightarrow$ the set of all real #'s is uncountable

Cantor Set | exists perfect sets in R^1 which contain no segment.

$$P = \bigcap_{n=1}^{\infty} E_n$$

a) $E_1 \supset E_2 \supset E_3 \supset \cdots$
b) E_n is the union of 2^n intervals each length 3^{-n}

P is compact
P is not empty (Thm 2.36)
Cantor sets show a uncountable set of measure zero

Connected Sets

separated: if both $A \cap \bar{B}$ & $\bar{A} \cap B$ are empty

set $E \subset X$ is connected if E is not a union of 2 emptysets.

Thm | A subset E of the real lines R^1 is connected iff it has the following property:
If $x \in E$, $y \in E$ and $x < z < y$, then $z \in E$.