Rudin Ch 4 Continuity

Continuity of Functions

Limits of Functions:

4.1 - Def — Let X and Y be metric spaces; suppose $E \subset X$, f maps E into Y, and p is a limit point of E. We write $f(x) \to q$ as $x \to p$ or

(1) $\lim_{x \to p} f(x) = q$ — if there is a point $q \in Y$ with property.

For every $\varepsilon > 0$ there is an $\delta > 0$ s.t.

(2) $d_Y(f(x), q) < \varepsilon$ — d_X, d_Y refer to distances in X and Y

for all points $x \in E$ for which

(3) $0 < d_X(x, p) < \delta$

Theorem - 4.2 — (4) $\lim_{x \to p} f(x) = q$

iff (5) $\lim_{n \to \infty} f(p_n) = q$

for every seq. $\{p_n\}$ in E s.t.

(6) $p_n \neq p$, $\lim_{n \to \infty} p_n = p$.

Corollary - If f has a limit at p, this limit is unique.

Theorem - 4.4 — $E \subset X$, a metric space, p is a limit point of E, f and g are complex functions on E and

$\lim_{x \to p} f(x) = A$ $\lim_{x \to p} g(x) = B$

then

a) $\lim_{x \to p} (f+g)(x) = A + B$ — if f and g map into $\mathbb{R}^k$

b) $\lim_{x \to p} (fg)(x) = AB$ — the (a) holds

c) $\lim_{x \to p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$ if $B \neq 0$ — (b') $\Rightarrow \lim_{x \to p} (f \cdot g)(x) = A \cdot B$

Continuous Functions

4.5 - Definition Suppose X and Y are metric spaces, $E \subset X$, $p \in E$, and f maps E into Y.
f is continuous at p if for $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t.

$$d_Y(f(x), f(p)) < \varepsilon \quad \text{for } \forall x \in E \text{ for which } d_X(x, p) < \delta$$

$\Rightarrow$ if f is continuous at every point of E, f is continuous on E

$\Rightarrow$ f has to be defined at point p to be continuous at p.

4.6 - Theorem If p is a limit point of E then, f is continuous at p iff $\lim_{x \to p} f(x) = f(p)$

4.7 - Theorem Suppose X, Y, Z are metric spaces, $E \subset X$, f maps E into Y, g maps range of f, $f(E)$, into Z, and h is the mapping E onto Z

$$h(x) = g(f(x)) \quad (x \in E)$$

If f is continuous at a point $p \in E$ and if g is continuous at the point $f(p)$, then h is continuous at p.

$\Rightarrow$ h is the composite of f and g written as $h = g \circ f$.

4.8 - Theorem - A mapping f of a metric space X into a metric space Y is continuous on X iff $f^{-1}(V)$ is open in X for every open set V in Y.

Corollary - A mapping f of a metric space X into a metric space Y is continuous iff $f^{-1}(C)$ is closed in X for every closed set C in Y.

4.9 - Theorem | Let f and g be complex continuous functions on a metric space X. Then $f+g$, fg and f/g are continuous on X.

4.10 - Theorem |

(a) Let $f_1, \dots f_k$ be real functions on a metric space X, and let f be the mapping of X into R^k defined by

$$(7) \quad f(x) = (f_1(x), \dots, f_k(x)) \quad (x \in X);$$

then f is continuous iff each of the functions $f_1, \dots, f_k$ is continuous.

(b) If f and g are continuous mappings of X into R^k, then $f+g$ and $f \cdot g$ are continuous on X.

⇒ functions $f_1, \dots, f_k$ are components of f.
⇒ $f+g$ is a mapping into R^k
⇒ $f \cdot g$ is a real function on X.

⟹ EXAMPLE 4.11 pg 88

⇒ Continuous mappings of one metric space into another is a better way to think about mappings of subsets.

Continuity And Compactness

4.13 Def - A mapping f of a set E into R^k is **bounded** if there is a real number M such that $|f(x)| \leq M$ for all $x \in E$.

4.14 - Theorem Suppose f is a continuous mapping of a compact metric space X into metric space Y. Then $f(X)$ is compact.

4.16 - Theorem If f is a continuous real function on a compact metric space X, and

(14) $M = \sup_{p \in X} f(p)$ $m = \inf_{p \in X} f(p)$

then there exists points $p, q \in X$ such that $f(p) = M$ $f(q) = m$

M = least upper bound, m = greatest lower bound.

$\Leftrightarrow$ There exist points p and q in X st. $f(q) \leq f(x) \leq f(p)$ for all $x \in X$. max @ p min @ q.

Theorem 4.15 - If f is a continuous mapping of a compact metric space X into R^k, then $f(X)$ is closed and bounded. Thus, f is bounded.

4.17 - Theorem f is a continuous 1-1 mapping of a compact metric space X onto metric space Y. Then, the inverse mapping f^{-1} defined on Y by

$$f^{-1}(f(x)) = x \quad (x \in X)$$

is a continuous mapping of Y onto X.

4.18 - Def | Let f be a mapping of a metric space X into a metric space Y. We say that f is uniformly continuous on X if for every $\varepsilon > 0$ there exists $\delta > 0$ s.t.

$$d_Y(f(p), f(q)) < \varepsilon$$

for all p and q in X for which $d_X(p, q) < \delta$

uniformly continuous is a property of a function on a set.

continuity can be defined at a single point.

4.19 - Theorem | Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X.

4.20 Theorem | Let E be a noncompact set in R^1. Then

(a) there exists a continuous function E which is not bounded.

(b) there exists a continuous and bounded function on E which has no maximum.

If, in addition, E is bounded, then.

(c) there exists a continuous function on E which is not uniformly continuous.

Continuity & Connectedness

[4.22 - Theorem] If f is a continuous mapping of a metric space X into a metric space Y, and if E is a connected subset of X, then f(E) is connected.

[4.23 - Theorem] let f be a continuous real function on the interval $[a, b]$. If $f(a) < f(b)$ and if c is a number that $f(a) < c < f(b)$, then there is a point $x \in (a, b)$ such that $f(x) = c$.

$\Rightarrow$ Converse is NOT TRUE.

DISCONTINUITES

if x is in the domain of f but f is not continuous at x we say "f is discontinuous at x" or "f has discontinuity @ x".

[Def - 4.25] f is defined on (a, b) x s.t. $a \leq x < b$

Right Hand limit $f(x+) = q$ if $f(t_n) \to q$ as $n \to \infty$ for all seq. $\{t_n\}$ in (x, b) s.t. $t_n \to x$

Left hand limit $f(x-)$ for $a < x \leq b$ for seq $\{t_n\}$ in (a, x)

Point x of (a, b) has a limit $\lim_{t \to x} f(t)$ iff

$$f(x+) = f(x-) = \lim_{t \to x} f(t)$$

[Def - 4.26]

2 Kinds of Discontinuities

Simple Discontinuity (First Kind)

if either $f(x+) \neq f(x-)$ $\Rightarrow$ $f(x)$ is immaterial

or if $f(x+) = f(x-) \neq f(x)$

otherwise the discontinuity is of Second Kind

Monotonic Functions

4.28 - Def | let f be real on (a,b)

f is monotonically increasing on (a,b)
if $a<x<y<b$ implies $f(x) \leq f(y)$

monotonically decreasing on (a,b) if $a<x<y<b$
implies $f(x) \geq f(y)$

4.29 Theorem | Let f be monotonically increasing
on (a,b). Then $f(x+)$ and $f(x-)$
exist at every point of x of (a,b)

$$\Rightarrow \sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t)$$

$\Rightarrow$ if $a<x<y<b$ then $f(x+) \leq f(y-)$

Corollary | Monotonic functions have no second kind
discontinuities

4.30 - Theorem | Let f be monotonic on (a,b).
Then the set of points (a,b) at which
f is discontinuous is at most countable.

if $f(x-) = f(x)$ $\forall (a,b)$ = continuous from the left

$f(x+) = f(x)$ $\forall (a,b)$ = continuous from the right.