Rudin Ch. 5 : Differentiation

The Derivative of a real function

5.1 Def : let f be defined on $[a,b]$. For any $x \in [a,b]$ form the quotient

(1) $$\phi(t) = \frac{f(t) - f(x)}{t - x} \quad (a < t < b,\ t \neq x)$$

(2) $$f'(x) = \lim_{t \to x} \phi(t)$$

$\Rightarrow$ if f is defined on (a,b) and if $a < x < b$, then $f'(x)$ is defined by (1), (2) but $f'(a)$, $f'(b)$ are not.

5.2 Thm | let f be defined on $[a,b]$. If f is differentiable at a point $x \in [a,b]$, then f is continuous at x.

$\Rightarrow$ converse NOT true

5.3 Thm | (a) $(f+g)'(x) = f'(x) + g'(x)$

(b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

(c) $$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$$

Every polynomial and rational functions (excluding denominator = 0) are differentiable.

Chain Rule 5.5 | f is continuous on $[a,b]$, $f'(x)$ exists at a point $x \in [a,b]$, g is defined on interval I, which contains the range of f, and g is differentiable at the point $f(x)$

if $h(t) = g(f(t))$ $(a \leq t \leq b)$

h is differentiable at x $\quad h'(x) = g'(f(x))f'(x)$

Mean Value Theorem

5.7 Def) Let f be a real function defined on metric space X. f has a local max at a point $p \in X$ if there exists $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in X$ with $d(p,q) < \delta$

$\Rightarrow$ local min defined similarly.

5.8 Thm) Let f be defined on $[a,b]$; if f has a local maximum at a point $x \in (a,b)$ and if $f'(x)$ exists, then $f'(x) = 0$.

$\Rightarrow$ min same statement.

5.9 Thm) If f and g are continuous real functions on $[a,b]$ which are differentiable in (a,b), then there is a point $x \in (a,b)$ at which

$$[f(b) - f(a)]\, g'(x) = [g(b) - g(a)]\, f'(x)$$

5.10 Thm) If f is a real continuous function on $[a,b]$ which is differentiable in (a,b), then there is a point $x \in (a,b)$ at which

$$f(b) - f(a) = (b-a) f'(x)$$

Thm 5.11) Suppose f is differentiable in (a,b)

(a) If $f'(x) \geq 0$ for $\forall x \in (a,b)$, then f is monotonically $\uparrow$

(b) If $f'(x) = 0$ for $\forall x \in (a,b)$, then f is constant

(c) if $f'(x) \leq 0$ for $\forall x \in (a,b)$, then f is monotonically $\downarrow$

The Continuity of Derivatives

5.12 Thm | Suppose f is a real differentiable function on $[a,b]$ and suppose $f'(a) < \lambda < f'(b)$

Then there is apoint $x \in (a,b)$ s.t. $f'(a) > f'(b)$

$\Rightarrow$ similar result for $f'(a) > f'(b)$

Corollary | if f is differentiable on $[a,b]$ then f' cannot have any simple discontinuities on $[a,b]$.

$\Rightarrow$ f' may have discontinities of the second kind

L'Hospital's Rule: f and g are real /differentiable in (a,b) and $g'(x) \neq 0$ for all $x \in (a,b)$, where $-\infty \leq a < b \leq +\infty$

suppose $\frac{f'(x)}{g'(x)} \to A$ as $x \to a$

if $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$

or if $g(x) \to +\infty$ as $x \to a$

then $\frac{f(x)}{g(x)} \to A$ as $x \to a$.

Derivatives of Higher Order.

In order for $f^{(n)}(x)$ to exist at a point x, $f^{(n-1)}(t)$ must exist in a neighborhood of x, and $f^{(n-1)}$ must be differentiable at x.

Since $f^{(n-1)}$ must exist in a neighborhood of x, $f^{(n-2)}$ must be differentiable in that neighborhood

Taylor's Theorem

Suppose f is a real function on $[a,b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a,b]$, $f^{(n)}(t)$ exists for every $t \in (a,b)$. Let α, β be distinct points of $[a,b]$ and define

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k$$

then there exists a point x between α \$ β s.t.

$$\rightarrow f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n$$

for $n=1 \iff$ mean Value theorem.

Shows that f can be approximated by a polynomial degree $n-1$

$f(\beta)$ allows us to estimate error if we know the bounds on $|f^{(n)}(x)|$